New just-infinite pro-p groups of finite width and subgroups of the Nottingham group

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Abstract

We use the action of the Nottingham group on the completion of its Lie algebra to construct new examples of just-infinite pro-p groups of finite width whose graded Lie algebras with respect to the lower central series are isomorphic to $\mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$. The groups constructed are non-linear and thus answer the question posed in [Ba]. The proof of nonlinearity uses a new description of the centralizers of elements of order p in the Nottingham group and the concept of Hausdorff dimension.

1 Introduction

A pro-p group G is said to be of finite width if all quotients $\gamma_n(G)/\gamma_{n+1}(G)$ are finite and their orders are uniformly bounded. The motivation for the study of these groups comes from two sources. On the one hand, pro-p groups of finite width form a natural generalization of the important and now well understood class of groups of finite coclass (see [LM] and [DDMS]). On the other hand, it is interesting to compare their structure to that of "narrow" objects in other algebraic categories (e.g. graded Lie algebras) where some definitive results have been obtained during the past two decades (see [ShZ]). The most ambitious project in the theory is the problem of classification of just-infinite pro-p groups of finite width (a profinite group is called just-infinite if it is infinite, but all of its proper homomorphic images are finite; these groups can be thought of as simple objects in the category of infinite profinite groups). All previously known examples of such groups can be divided into four types (see [LM], [Sh] and [KLP] for more details):

1. p-adic analytic groups,

2. $\mathbb{F}_p[[t]]$-analytic (linear) groups,

3. the Nottingham groups $\mathcal{N}(\mathbb{F}_q)$, where $q = p^n$, and some of their index subgroups (see [BK] and [F]),

4. branch groups.

In this paper we introduce and study a new family of just-infinite pro-p groups of finite width, defined as certain subgroups of the Nottingham group $\mathcal{N}(\mathbb{F}_p)$. Recall that $\mathcal{N}(\mathbb{F}_p)$ is the group of automorphisms of $\mathbb{F}_p[[t]]$ acting trivially on $(t)/(t^2)$. Assume that $p > 2$, and let r and s be positive integers such that $0 < r < p^s/2$ and $p \nmid r$. Let $\mathcal{Q} = \mathcal{Q}^1(s, r)$ be the subgroup of $\mathcal{N}(\mathbb{F}_p)$ which consists of the elements of the form

$$t \mapsto \sqrt[r]{\frac{at^r + b}{ct^r + d}} \text{ for some } a, b, c, d \in \mathbb{F}_p[[t^{p^s}]] \text{ s.t. } a_0 = d_0 = 1 \text{ and } c_0 = b_0 = 0$$

(here a_0, b_0, c_0, d_0 are the constant terms of the corresponding elements).

The above definition resembles that of the first congruence subgroup of $PGL_2(\mathbb{F}_p[[t]])$ which we denote by $PGL_2^1(\mathbb{F}_p[[t]])$. Note that since $p > 2$, $PGL_2^1(\mathbb{F}_p[[t]]) \cong SL_2^1(\mathbb{F}_p[[t]])$. As we will show in the present paper (see Sections 4 and 8), there are indeed a lot of similarities between the groups $\mathcal{Q}^1(s, r)$ and $SL_2^1(\mathbb{F}_p[[t]])$. In particular, the graded Lie algebras of these groups with respect to the lower central series are isomorphic to each other.

Theorem 1.1. *Let p, s, r be as above, and let $\mathcal{Q} = \mathcal{Q}^1(s, r)$. The Lie algebra of $\mathcal{Q}$ with respect to the lower central series is isomorphic to $\mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$ as a restricted Lie algebra. Therefore $\mathcal{Q}$ is a hereditarily just-infinite pro-p group of finite width.*

Remark. The Lie algebra of $\mathcal{Q}$ has a natural restricted structure because the lower central series of $\mathcal{Q}$ coincides with the Zassenhaus filtration.

On the other hand, the groups $\{\mathcal{Q}^1(s, r)\}$ are not linear. More precisely, we have the following.

Theorem 1.2. *Let $\mathcal{Q} = \mathcal{Q}^1(s, r)$.*
1. $\mathcal{Q}$ is not linear over a pro-p ring (as a topological group).
2. If $s = 1$, then $\mathcal{Q}$ is not linear over any field (as an abstract group).

Finally, it is natural to ask whether the groups $\{\mathcal{Q}^1(s, r)\}$ are pairwise non-commensurable (or at least non-isomorphic). We do not know the answer to that question, but we can show that there are infinitely many pairwise non-commensurable groups in this family.

Theorem 1.3. *The groups $\{\mathcal{Q}^1(s,1)\}_{s=1}^{\infty}$ are pairwise non-commensurable.*

Remarks. 1) Barnea and Klopsch constructed another family of just-infinite pro-p groups of finite width consisting of "natural" subgroups of $\mathcal{N}(\mathbb{F}_p)$ (see [BK]). Their groups can be thought of as the "upper-triangular" subgroups of the groups $\{\mathcal{Q}^1(s,1)\}_{s=1}^{\infty}$ (see Section 8).

2)In the terminology of Barnea (see [Ba]), a pro-p group G is a $\mathfrak{g}$-group, where $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{F}_p$, if the Lie algebra of G with respect to some filtration is isomorphic to $\mathfrak{g}\otimes t\mathbb{F}_p[t]$ as a graded Lie algebra. If $\mathfrak{g}$ is simple, G is called Lie simple; such groups are always just-infinite of finite width. Typical examples of Lie simple groups are $SL_n^1(\mathbb{Z}_p)$ and $SL_n^1(\mathbb{F}_p[[t]])$, $p\nmid n$; in both cases $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F}_p)$. Barnea asks whether there exists a nonlinear $\mathfrak{sl}_n(\mathbb{F}_p)$-group ([Ba, Problem 1]) and, more generally, whether the Lie algebra of such group can be isomorphic to $\mathfrak{sl}_n(\mathbb{F}_p)\otimes t\mathbb{F}_p[t]$ as a restricted Lie algebra ([Ba, Problem 2]). Thus our results give a positive answer to both questions for $n=2$. We are grateful to the referee for drawing our attention to the second of Barnea's questions.

Before proceeding, we remark that the groups $\{\mathcal{Q}^1(s,r)\}$ can be defined in a more conceptual way. Let $\mathrm{Lie}(\mathcal{N})$ be the completion of the graded Lie algebra of the Nottingham group $\mathcal{N}$ with respect to the congruence filtration $\{\mathcal{N}_n\}$ where $\mathcal{N}_n$ consists of automorphisms acting trivially on $(t)/(t^{n+1})$. We will show that $\mathcal{N}$ has a natural action on $\mathrm{Lie}(\mathcal{N})$. On the other hand, it is well known that $\mathrm{Lie}(\mathcal{N})$ has a family of subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{F}_p)\otimes t\mathbb{F}_p[[t]]$. It turns out that each of the groups $\{\mathcal{Q}^1(s,r)\}$ is a finite index subgroup of the stabilizer of one of these subalgebras under the action of $\mathcal{N}$. The whole stabilizer will be denoted by $\mathcal{Q}(s,r)$; its explicit description is very similar to that of $\mathcal{Q}^1(s,r)$ (see Section 3), and the "corresponding" matrix group is a Sylow pro-p subgroup of $SL_2(\mathbb{F}_p[[t]])$.

Nonlinearity of the groups $\{\mathcal{Q}(s,r)\}$ follows from our analysis of the subgroup structure of the Nottingham group which is discussed below. In order to state our results we need to recall the concept of Hausdorff dimension of a subset of a profinite group which will play a crucial role in this paper. If G is a profinite group with fixed filtration $\{G_n\}$, one introduces certain metric on G (which depends on the choice of the filtration) and given a closed subset H of G, one defines $\mathrm{Hdim}_G(H)$ to be the Hausdorff dimension of H with respect to the metric induced from G. If H is a subgroup of G, its dimension can be computed by the formula

$$\mathrm{Hdim}_G H=\liminf_{n\to\infty}\frac{\log|HG_n:G_n|}{\log|G:G_n|}=\liminf_{n\to\infty}\frac{\log|H:H\cap G_n|}{\log|G:G_n|}.$$

If the limit of the above expression exists, we say that H has pure Hausdorff dimension. Finally, if H has the same Hausdorff dimension with respect to any filtration of G, we say that H has absolute Hausdorff dimension. For more details see [BSh] or Section 5 of this paper.

We begin by considering the centralizers of elements of order p in $\mathcal{N}(\mathbb{F}_p)$. Elements of order p and their centralizers have already been studied by Klopsch in connection with the automorphism group of the Nottingham group. In particular, he proves the following (see [K] and [BK]).

Theorem 1.4 (Klopsch). *1. The elements* $\{f_n(\lambda) = t(1-\lambda t^n)^{-1/n} \mid n \in \mathbb{N}, p \nmid n, \lambda \in \mathbb{F}_p{}^*\}$ *form a complete set of representatives for the conjugacy classes of elements of order* p *in* $\mathcal{N}(\mathbb{F}_p)$.
2. The centralizer of an element of order p *has absolute Hausdorff dimension* $1/p$.

Actually, Klopsch describes these centralizers explicitly, and part 2 of the above theorem is an easy consequence of this description (see [BK]). In this paper we obtain a closed formula for the centralizers of the elements $\{f_n\}$ above. Combined with Theorem 1.4, this formula enables us to prove the following result:

Theorem 1.5. *Let* $G = \mathcal{N}(\mathbb{F}_p)$ *and let* $g \in G$ *be an element of order* p. *Then the group* $\mathrm{Cent}_G(g)/\langle g\rangle$ *is isomorphic to an open subgroup of* G.

One of the applications of Theorem 1.5 is an alternative proof of non-linearity of $\mathcal{N}(\mathbb{F}_p)$ as an abstract group (for the other proof of this fact see [Ca2]). Moreover, using our explicit formulas, we are able to prove a statement analogous to Theorem 1.5 for the groups $\{\mathcal{Q}(1,r)\}$. This statement is weaker but still implies that these groups are non-linear as abstract groups.

Now we will explain why all the groups $\{\mathcal{Q}(s,r)\}$ are not linear over a pro-p ring. The key difference between the Nottingham group and linear pro-p groups lies in the sizes of abelian subgroups as can be seen from the following theorems.

Theorem 1.6. *Let* G *be a just-infinite pro-p group which is linear over a pro-p ring. Then* G *has an open subgroup* H *which either is torsion free or contains a subgroup* K *isomorphic to* $(\mathbb{F}_p[[t]],+)$ *such that* K *has positive Hausdorff dimension with respect to some finite width filtration of* H.

The information about abelian subgroups of the Nottingham group $\mathcal{N}$ is provided by the following theorem of Wintenberger [Wi, Theorem 4.1].

Theorem 1.7 (Wintenberger). *Let H be an abelian subgroup of the Nottingham group $\mathcal{N} = \mathcal{N}(\mathbb{F}_p)$ and let $I = \{i \in \mathbb{N} \mid H \cap \mathcal{N}_i \neq H \cap \mathcal{N}_{i+1}\}$, where $\{\mathcal{N}_i\}$ is the congruence filtration of $\mathcal{N}$. Let $i_1 < i_2 < \dots$ be the elements of I listed in increasing order. Then $i_{n+1} \equiv i_n \mod p^n$ for each $n \in \mathbb{N}$.*

Here is an interesting consequence of (the proof of) Theorem 1.5 and Theorem 1.7 which generalizes part 2 of Theorem 1.4.

Theorem 1.8. *Let H be a subgroup of $G = \mathcal{N}(\mathbb{F}_p)$. Then the centralizer of H in G has absolute Hausdorff dimension $\dfrac{1}{|H|}$ (we do not assume H is finite). In particular, an abelian subgroup of the Nottingham group has absolute Hausdorff dimension zero.*

Corollary 1.9. *Let H be a hereditarily rigid just-infinite pro-p group which is not virtually torsion free. Suppose that there exists an embedding $\Phi : H \to G = \mathcal{N}(\mathbb{F}_p)$ such that $\mathrm{Hdim}_G \Phi(H) > 0$. Then H is not linear over a pro-p ring.*

Remarks.
1) The notion of a rigid group which appears in the statement of the corollary is a slight variation of the concept of finite obliquity (precise definition is given in Section 5). The assumption of H being hereditarily rigid is needed to ensure that certain subgroups of H have absolute Hausdorff dimension. It is easy to show that all the groups $\{\mathcal{Q}(s,r)\}$ satisfy this requirement.

2) The proof of Corollary 1.9 uses Theorem 1.8 only in the special case when H is torsion which in turn doesn't rely on Wintenberger's theorem. We believe that a more detailed analysis of the structure of linear groups combined with Theorem 1.7 should yield the following generalization of Corollary 1.9.

Conjecture 1.10. *A just-infinite pro-p group which is linear over a pro-p ring can't be embedded into the Nottingham group as a subgroup of positive Hausdorff dimension.*

Finally, we remark that the Hausdorff dimension of the group $\mathcal{Q}(s,r)$ in $\mathcal{N}$ is equal to $3/p^s$. The number $3/p$ is a previously unknown point in the Hausdorff spectrum of the Nottingham group. Combining this observation with the results of [BK], we conclude that the only part of the spectrum of $\mathcal{N}$ which is still undetermined lies in the interval $(1/p, 2/p)$. Moreover, the study of the subgroup structure of the groups $\{\mathcal{Q}(1,r)\}$ may help to fill this gap. In particular, the subgroups discussed in Section 8 of this paper yield another new point in the spectrum of $\mathcal{N}$, namely $\{3/2p\}$.

Organization. The paper is organized as follows. Section 2 contains preliminaries on the Nottingham group and its Lie algebra. In Section 3 we construct the groups $\{\mathcal{Q}(s,r)\}$. This is followed by the proof of Theorem 1.1 in Section 4. In Section 5 we review some facts about Hausdorff dimension and define rigid groups. In Section 6 we compute the centralizers of elements of order p in $\mathcal{N}(\mathbb{F}_p)$ and prove Theorems 1.5 and 1.8. Linearity criteria are established in Section 7. We use them to prove Theorem 1.2. In Section 8 we discuss the subgroup structure of the groups $\{\mathcal{Q}(s,r)\}$. In section 9 we obtain a partial classification of some natural subgroups of $\mathcal{N}$ up to commensurability. In particular, we prove Theorem 1.3. Finally, in Section 10 we pose several questions which arose from the results of this paper.

Preliminaries and Notation. Throughout the paper, groups are assumed to be topological unless indicated otherwise; by a subgroup of a topological group we always mean a closed subgroup. As usual $(g,h) = g^{-1}h^{-1}gh$ will stand for the commutator of elements g and h. If A, B are subgroups of G, (A,B) is defined as the subgroup generated by $\{(a,b) \mid a \in A, b \in B\}$. The n^{th} term of the lower central series of G will be denoted by $\gamma_n G$. We write $H \lhd G$ if H is a normal subgroup of G and $H \underset{o}{\lhd} G$ if H is also open.

Two groups are called commensurable if they have isomorphic subgroups of finite index. We say that a group G is hereditarily (P), where (P) is some group-theoretic property, if every finite index subgroup of G satisfies (P).

By a filtration of a pro-p group G we mean a descending chain of open normal subgroups $G = G_1 \supseteq G_2 \supseteq \ldots$ which form a base of neighborhoods of identity. A filtration is called central if $[G_i, G_j] \subseteq G_{i+j}$, p-central if in addition $G_i^p \subseteq G_{i+1}$, and restricted if $G_i^p \subseteq G_{pi}$. The lower central series $\{\gamma_n G\}$ is a central filtration whenever $\gamma_n G$ is open in G for all n. The Zassenhaus filtration $\{\Omega_n G\}$, where $\Omega_n G = \prod\limits_{m \cdot p^i \geq n} (\gamma_m G)^{p^i}$, is an example of a restricted filtration. Finally, we say that a filtration is of finite width if the orders of the quotients G_n/G_{n+1} are uniformly bounded.

Given a central filtration $\{G_n\}$ of G, we can construct the associated graded Lie ring $L(G) = \bigoplus\limits_{n=1}^{\infty} L_n$, where $L_n = G_n/G_{n+1}$, with bracket defined by $[aG_{n+1}, bG_{m+1}] = (a,b)G_{n+m+1}$. With each subgroup H of G we associate a Lie subring $L_G(H) = \bigoplus\limits_{n=1}^{\infty} (H \cap G_n)G_{n+1}/G_{n+1} \subseteq L(G)$. If the filtration $\{G_n\}$ is p-central, $L(G)$ becomes a Lie algebra over $\mathbb{F}_p$, and if $\{G_n\}$ is restricted, then $L(G)$ has the structure of a restricted Lie algebra where $[aG_{n+1}]^p = [a^p G_{pn+1}]$. Sometimes it'll be more convenient for us to work with the completed Lie algebra $\overline{L}(G) = \prod_{n=1}^{\infty} L_n$ considered as a filtered algebra with filtration $\{L^n\}$, where $L^n = \prod_{i=n}^{\infty} L_i$.

Finally, we say that $g \in G$ has degree n with respect to some central filtration $\{G_n\}$ if $g \in G_n \backslash G_{n+1}$ and define the leading term of g to be LT $g = gG_{n+1} \in L_n$. Thus LT is just the usual map from G to $L(G)$.

Throughout the paper $p > 2$ will be a fixed prime number.

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2 The Nottingham group and its Lie algebra

We begin this section by recalling some basic properties of the Nottingham group. For more details the reader is referred to [Ca2].

Given a commutative ring R, the Nottingham group $\mathcal{N}(R)$ is defined as the group of R- linear automorphisms of the ring $R[[t]]$, acting trivially on $(t)/(t^2)$. It can also be thought of as the group of formal power series $\{t(1+r_1t+r_2t^2+\dots) \mid r_i \in R\}$ under substitution. In this paper we restrict ourselves to the case $R = \mathbb{F}_p$ and we will often write $\mathcal{N}$ instead of $\mathcal{N}(\mathbb{F}_p)$.

To avoid potential ambiguity caused by two possible interpretations of elements of the Nottingham group (automorphisms versus power series under substitution) we fix the following notation: if φ and ψ are two power series, then

$\varphi \cdot \psi$ denotes the product in the ring $\mathbb{F}_p[[t]]$,

$\varphi \circ \psi$ denotes the composition of power series (if it is defined),

$[\varphi]$ will denote the element of Aut $(\mathbb{F}_p[[t]])$ which sends t to φ.

Note that multiplication in $\mathcal{N}(\mathbb{F}_p)$ corresponds to composition of power series in reverse order: $[\varphi]\cdot[\psi] = [\psi\circ\varphi]$. We also agree to denote the coefficient of t^i of an element $r \in \mathbb{F}_p[[t]]$ by r_i, e.g. if $r = 1+2t+3t^2+\dots$, then $r_2 = 3$.

$\mathcal{N} = \mathcal{N}(\mathbb{F}_p)$ is a finitely generated pro-p group with some remarkable properties. On the one hand, $\mathcal{N}$ has a lot of similarities with $\mathbb{F}_p[[t]]$-analytic groups: it is hereditarily just-infinite, has finite width, finite obliquity, and finite lower rank; it also has the same subgroup growth type as $\mathbb{F}_p[[t]]$-perfect groups (see [LSh]). On the other hand, it is not linear and moreover, by a theorem of Camina ([Ca1]) every countably based pro-p group can be embedded as a closed subgroup of $\mathcal{N}$. Many of the above "narrowness" properties of the Nottingham group can be established by considering its graded Lie algebra $L = L(\mathcal{N}) = \bigoplus_{n=1}^{\infty} \mathcal{N}_n/\mathcal{N}_{n+1}$ with respect to the congruence filtration $\{\mathcal{N}_n\}$, where $\mathcal{N}_n = \{t\,(1 + a_n t^n + a_{n+1}t^{n+1} + \dots)\}$. It is well known that $L(\mathcal{N})$ is isomorphic to the positive part of the Witt algebra $W^+ = \mathrm{Der}^{\,+}\mathbb{F}_p[t] = \bigoplus_{i=1}^{\infty} \mathbb{F}_p\, e_i$, where $e_i = t^{i+1}\partial_t$ and $[e_i, e_j] = (j-i)e_{i+j}$. An isomorphism between the two algebras is given by the map $\mathrm{LT}\,([t(1+t^n)]) \mapsto e_n$.

Let $\mathrm{Lie}(\mathcal{N})$ denote the completion of $L(\mathcal{N})$ with respect to the filtration $\{L^n\}$, where $L^n = \bigoplus_{k=n}^{\infty} \mathbb{F}_p e_k$. This Lie algebra has a different "interpretation", which will give us a better correspondence between subgroups of $\mathcal{N}$ and subalgebras of $\mathrm{Lie}(\mathcal{N})$. It comes from the following easy observation: if R is a pronilpotent ring, then the group $G = \mathrm{Aut}\,(R)$ has a natural action on the Lie algebra $\mathrm{Der}^{\,+}(R)$ of pronilpotent derivations of R defined as follows: if $g \in G$, $d \in \mathrm{Der}\,(R)$ and $r \in R$, set

$$(g\,d)(r) = gdg^{-1}r.$$

In the case $R = \mathbb{F}_p[[t]]$ we have a natural isomorphism $\mathrm{Der}^{\,+}(R) \cong \mathrm{Lie}(\mathcal{N})$, and the corresponding action of the Nottingham group on $\mathrm{Lie}(\mathcal{N})$ is easily seen to be faithful. One can check that the action $\pi : \mathcal{N} \to \mathrm{Aut}\,(\mathrm{Lie}(\mathcal{N}))$ is given by the formula

$$\pi g\ (p(t)\partial_t) = \frac{p(g(t))}{g'(t)}\partial_t.$$

where $g'(t)$ if the formal derivative of the power series $g(t)$. It follows from the above formula that if $u \in L(\mathcal{N}) \subset \mathrm{Lie}(\mathcal{N})$ is homogeneous and $g \in \mathcal{N}$, then $\pi g(u) = u + [\mathrm{LT}\, g, u] + $higher order terms. Notice that the action of $\mathcal{N}$ does not preserve the grading on $\mathrm{Lie}(\mathcal{N})$, but it does preserve the filtration $\{L^n\} = \{\prod_{i\geq n} \mathbb{F}_p e_i\}$. Actually, when $p > 3$, one can show that $\mathcal{N}$ is equal to the group of all filtered automorphisms of $\mathrm{Lie}(\mathcal{N})$ which act trivially on each quotient L^n/L^{n+1}. This is an easier analogue of a theorem of Klopsch [K] which asserts that all normalized automorphisms of $\mathcal{N}(\mathbb{F}_p)$ are inner.

The action of $\mathcal{N}$ on $\mathrm{Lie}(\mathcal{N})$ can be used to construct explicit examples of subgroups of $\mathcal{N}$. For example, given a subalgebra M of $\mathrm{Lie}(\mathcal{N})$, we may consider the subgroup $\mathrm{Stab}_{\mathcal{N}}(M) = \{\varphi \in \mathcal{N} \mid \varphi(M) \subseteq M\}$.

3 Construction of the groups $\mathcal{Q}(s,r)$

Let r and s be positive integers such that $0 < r < p^s/2$ and $p \nmid r$. We also set $q = p^s$. Consider the subalgebra $\mathfrak{q} = \mathfrak{q}(s,r) = \prod_{n \equiv 0, \pm r \mod p^s} \mathbb{F}_p e_n$ of $L = \mathrm{Lie}(\mathcal{N})$ which is commensurable to $\mathfrak{sl}_2(\mathbb{F}_p[[t]])$. It turns out that the stabilizer of this subalgebra coincides with the group $\mathcal{Q}(s,r)$ announced in the introduction.

Proposition 3.1. *Let $\mathfrak{q} = \mathfrak{q}(s,r)$ and let $\mathcal{Q} = \mathcal{Q}(s,r)$ be the subgroup of $\mathcal{N}$ which consists of the elements of the form*
$t \mapsto \sqrt[r]{\dfrac{at^r+b}{ct^r+d}}$ *for some $a,b,c,d \in \mathbb{F}_p[[t^{p^s}]]$ s.t. $a_0 = d_0 = 1$ and $b_0 = 0$.*

Then $\mathcal{Q}$ has the following properties:
1) $\mathcal{Q} = \mathrm{Stab}_{\mathcal{N}}(\mathfrak{q}) = \{\varphi \in \mathcal{N} \mid \varphi(\mathfrak{q}) = \mathfrak{q}\}$.
2) $\mathrm{Lie}_{\mathcal{N}}(\mathcal{Q}) = \mathfrak{q}$.

Proof. First, let's check that $\mathcal{Q}$ is a subgroup:

$$\left[\sqrt[r]{\frac{a_1t^r+b_1}{c_1t^r+d_1}}\right] \cdot \left[\sqrt[r]{\frac{a_2t^r+b_2}{c_2t^r+d_2}}\right](t) = \left(\frac{\tilde{a}_2 \cdot \dfrac{a_1t^r+b_1}{c_1t^r+d_1} + \tilde{b}_2}{\tilde{c}_2 \cdot \dfrac{a_1t^r+b_1}{c_1t^r+d_1} + \tilde{d}_2}\right)^{1/r} =$$

$$\sqrt[r]{\frac{(\tilde{a}_2a_1 + \tilde{b}_2c_1)t^r + (\tilde{a}_2b_1 + \tilde{b}_2d_1)}{(\tilde{c}_2a_1 + \tilde{d}_2c_1)t^r + (\tilde{c}_2b_1 + \tilde{d}_2d_1)}}, \text{ where } \tilde{a}_2 = a_2 \circ \sqrt[r]{\frac{a_1t^r+b_1}{c_1t^r+d_1}} \text{ etc.}$$

Since $a_2 \in \mathbb{F}_p[[t^q]]$, we have $a_2 \circ w \in \mathbb{F}_p[[t^q]]$ for any $w \in \mathbb{F}_p[[t]]$ whence $\mathcal{Q}$ is semigroup. It is also easy to see that $\mathcal{Q}$ is a closed subset of $\mathcal{N}$. Therefore, $\mathcal{Q}$ is a subgroup, since in a pro-p group g^{-1} is a limit of positive powers of g.
Now we will show that $\mathcal{Q} \subseteq K$, where $K = \mathrm{Stab}_{\mathcal{N}}(\mathfrak{q})$. We have:

$$\left[\sqrt[r]{\frac{at^r+b}{ct^r+d}}\right](e_i) = \left[\sqrt[r]{\frac{at^r+b}{ct^r+d}}\right](t^{i+1}\partial_t) = \frac{\left(\dfrac{at^r+b}{ct^r+d}\right)^{(i+1)/r}}{\left(\sqrt[r]{\dfrac{at^r+b}{ct^r+d}}\right)'}\partial_t =$$

$$\left(\frac{at^r+b}{ct^r+d}\right)^{(i+1)/r+1-1/r} \frac{(ct^r+d)^2}{t^{r-1}(ad-bc)}\partial_t = \frac{(at^r+b)^{i/r+1}}{(ct^r+d)^{i/r-1}} \frac{t^{1-r}}{ad-bc}\,\partial_t$$

(note that $a' = b' = c' = d' = 0$).

Now if $i = qk - r$, then $\left[\sqrt[r]{\dfrac{at^r + b}{ct^r + d}}\right](e_i) = (ct^r + d)^2 \cdot t^{1-r} \cdot R$,

where $R \in \mathbb{F}_p[[t^q]]$. Therefore the right hand side is of the form

$$\sum_{n \equiv 1, 1+r, 1-r \mod q} \lambda_n t^n \partial_t = \sum_{n \equiv 0, \pm r \mod q} \lambda_{n+1} e_n \in \mathfrak{q}.$$

Cases $i = qk$ and $i = qk + r$ are treated in a similar way.

Now we claim that $L_{\mathcal{N}}(\mathcal{Q}) \supseteq \mathfrak{q}$. This is clear since

$$\mathrm{LT}\left[\sqrt[r]{t^r + t^{qk}}\right] = \frac{1}{r} e_{qk-r}, \ \mathrm{LT}\left[t(1 + t^{qk})\right] = e_{qk}, \ \mathrm{LT}\left[\frac{t}{\sqrt[r]{1 - t^{qk+r}}}\right] = \frac{1}{r} e_{qk+r}.$$

Finally, we show that $\mathfrak{q} \supseteq L_{\mathcal{N}}(K)$. Choose any $\varphi \in K$ and let $u = \mathrm{LT}\ \varphi$. Since for any homogeneous $x \in L$ we have $\varphi(x) = x + [u, x] +$ elements of higher degree, it follows that $[u, \mathfrak{q}] \subseteq \mathfrak{q}$, whence $u \in \mathfrak{q}$ (because $\mathfrak{q}$ is clearly self-normalizing). Thus all leading terms of elements of K lie in $\mathfrak{q}$, which implies $L_{\mathcal{N}}(K) \subseteq \mathfrak{q}$. Therefore we have shown that $L_{\mathcal{N}}(K) \subseteq \mathfrak{q} \subseteq L_{\mathcal{N}}(\mathcal{Q})$ and $\mathcal{Q} \subseteq K$. These inclusions imply that $K = \mathcal{Q}$ and $L_{\mathcal{N}}(K) = \mathfrak{q} = L_{\mathcal{N}}(\mathcal{Q})$. $\square$

Remarks.

1. The group $\mathcal{Q}^1(s, r)$ is a subgroup of $\mathcal{Q}(s, r)$ of index p.
2. Two expressions $\left[\sqrt[r]{\dfrac{a_1 t^r + b_1}{c_1 t^r + d_1}}\right]$ and $\left[\sqrt[r]{\dfrac{a_2 t^r + b_2}{c_2 t^r + d_2}}\right]$ represent the same element of $\mathcal{Q}(s, r)$ if and only if there exists $\lambda \in \mathbb{F}_p((t^q))$ such that $a_2 = \lambda a_1$, $b_2 = \lambda b_1 \ldots$. Since $p \neq 2$, it is easy to see that each element of $\mathcal{Q}(s, r)$ has the unique presentation in the form $\left[\sqrt[r]{\dfrac{at^r + b}{ct^r + d}}\right]$ where $a \equiv d \equiv 1 \mod t\mathbb{F}_p[[t]]$ and $ad - bc = 1$. This fact will be used in Section 4.
3. If r is any positive number not divisible by p, we can define

$$\mathcal{Q}(s, r) = \{t \mapsto \sqrt[r]{\frac{at^r + b}{ct^r + d}} \text{ for some } a, b, c, d \in \mathbb{F}_p((t^{p^s}))\} \cap \mathcal{N}(\mathbb{F}_p).$$

If $r < p^s/2$, this definition is consistent with the original one. It is also easy to see that $\mathcal{Q}(s, r) = \mathcal{Q}(s, r')$ when $r' \equiv \pm r \mod p^s$, and therefore this generalization doesn't yield new groups. However, it allows us to represent every element of $\mathcal{Q}(s, r)$ in many different ways, which will be useful at times.

4 The groups $\mathcal{Q}^1(s,r)$ as deformations of $SL_2^1(\mathbb{F}_p[[t]])$

As mentioned in the introduction, the groups $\{\mathcal{Q}^1(s,r)\}$ can be thought of as deformations of $SL_2^1(\mathbb{F}_p[[t]])$. In fact there is a natural bijective map $\varphi_{s,r} : SL_2^1(\mathbb{F}_p[[t]]) \to \mathcal{Q}^1(s,r)$ given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left[\sqrt[r]{\frac{(a\circ t^q)\cdot t^r + (c\circ t^q)}{(b\circ t^q)\cdot t^r + (d\circ t^q)}}\right] = \left[\sqrt[r]{\frac{a^q t^r + c^q}{b^q t^r + d^q}}\right], \text{ where } q = p^s.$$

Notice that the map $\varphi_{s,r}$ "transposes matrices". This is due to our conventions about multiplication in the Nottingham group.

Remark. Under the same map the group $\mathcal{Q}(s,r)$ corresponds to a Sylow pro-p subgroup of $SL_2(\mathbb{F}_p[[t]])$.

Some useful properties of the map $\varphi_{s,r}$ are stated in the following proposition.

Proposition 4.1. *Let $G = SL_2^1(\mathbb{F}_p[[t]])$, $H = \mathcal{Q}^1(s,r)$ and let $\varphi = \varphi_{s,r}$ be as above. Let $G_n = SL_2^n(\mathbb{F}_p[[t]])$ be the n^{th} congruence subgroup of G and set $H_n = \varphi(G_n)$.*

1) $H_n = H \cap \mathcal{N}_{qn-r}$.

2) The groups $\{H_n\}$ form a central filtration of H.

3) If $g_1, g_2 \in G$ with $g_1 - g_2 \in t^n M_2(\mathbb{F}_p[[t]])$, then $\varphi(g_1) \equiv \varphi(g_2) \mod H_n$.

Proof. 1) The inclusion $H_n \subseteq H \cap \mathcal{N}_{qn-r}$ is obvious. Now let $g \in \operatorname{Im}\varphi \cap \mathcal{N}_{qn-r}$. Then there exist $a,b,c,d \in \mathbb{F}_p[[t]]$ with $ad - bc = 1$, $a_0 = d_0 = 1$ and $b_0 = c_0 = 0$ such that

$$g(t) = \sqrt[r]{\frac{a^q t^r + b^q}{c^q t^q + d^q}} = t\cdot(a/d)^{q/r}\cdot\left(\frac{1+(b/a)^q t^{-r}}{1+(c/d)^q t^r}\right)^{1/r}.$$

We have $b/a = \alpha t^{n_1} + \ldots,\ c/d = \beta t^{n_2} + \ldots,\ a/d = 1 + \gamma t^{n_3} + \ldots$ for some $n_1, n_2, n_3 \in \mathbb{N}$ and $\alpha, \beta, \gamma \neq 0$. Now $g(t) \equiv t(1 + \frac{\alpha}{r} t^{qn_1 - r} + \frac{\gamma}{r} t^{qn_3} - \frac{\beta}{r} t^{qn_2 + r}) \mod t^{N+1}\mathbb{F}_p[[t]]$, where $N = \min(qn_1 - r, qn_3, qn_2 + r)$. On the other hand we know that $g(t) \equiv t \mod t^{qn-r+1}\mathbb{F}_p[[t]]$, whence $n_1, n_2, n_3 \geq n$. Therefore $b, c \in t^n\mathbb{F}_p[[t]]$ and $a - d \in t^n\mathbb{F}_p[[t]]$. Since $ad - bc = 1$, it follows that $a-1, d-1 \in t^n\mathbb{F}_p[[t]]$, whence $g \in \varphi(G_n)$.

2) Set $\mathcal{Q}_n = H \cap \mathcal{N}_n$. Let $g_n = \sqrt[r]{t^r + t^{qn}} \in \mathcal{Q}_{qn-r} \backslash \mathcal{Q}_{qn}$. It follows from the results of the previous section that the group $\mathcal{Q}_{qn-r}/\mathcal{Q}_{qn}$ is elementary cyclic whence $\mathcal{Q}_{qn-r} = \langle g_n \rangle \mathcal{Q}_{qn}$. Therefore,

$$(H_n, H_m) = (\mathcal{Q}_{qn-r}, \mathcal{Q}_{qm-r}) \subseteq (\langle g_n\rangle, \langle g_m\rangle)(\langle g_n\rangle, \mathcal{Q}_{qm})(\langle g_m\rangle, \mathcal{Q}_{qn})(\mathcal{Q}_{qm}, \mathcal{Q}_{qn}).$$

The last three factors clearly lie in $\mathcal{Q}_{q(n+m)-r} = H_{n+m}$. To finish the proof it remains to check that $(g_n, g_m) \in \mathcal{N}_{q(n+m)-r}$. We have

$$g_{qm-r} \cdot g_{qn-r}(t) = \sqrt[r]{t^r + t^{qn}} \circ \sqrt[r]{t^r + t^{qm}} = \sqrt[r]{t^r + t^{qm} + t^{qn}\left(1 + t^{qm-r}\right)^{qn/r}} \equiv$$
$$t\sqrt[r]{1 + t^{qm-r} + t^{qn-r}\left(1 + t^{q^2m-qr}\right)^{n/r}} \equiv \sqrt[r]{t^r + t^{qm} + t^{qn}} \mod t^N \mathbb{F}_p[[t]].$$

where $N = 1 + qn - r + q^2m - qr = qn + qm - r + 1 + q((q-1)m - r) \geq qn + qm - r + 1$. The last expression is symmetric in m and n, whence $g_{qn-r} \cdot g_{qm-r}(t) - g_{qm-r} \cdot g_{qn-r}(t) \in t^{qm+qn-r+1}\mathbb{F}_p[[t]]$. Therefore, $g_{qm-r} \cdot g_{qn-r} \equiv g_{qn-r} \cdot g_{qm-r} \mod \mathcal{N}_{q(n+m)-r}$, and we are done.

3) If $g_1, g_2 \in G$ with $g_1 - g_2 \in t^n M_2(\mathbb{F}_p[[t]])$, then clearly $\varphi(g_1)(t) - \varphi(g_2)(t) \in t^{qn-r+1}\mathbb{F}_p[[t]]$, whence $\varphi(g_1) \equiv \varphi(g_2) \mod \mathcal{Q}_{qn-r}$.

□

Our next goal is to show that $\varphi_{s,r}$ induces an isomorphism of the graded Lie algebras of $SL_2^1(\mathbb{F}_p[[t]])$ and $\mathcal{Q}^1(s,r)$ with respect to the lower central series.

Definition. Let G and H be pro-p groups with fixed filtrations $\{G_n\}$ and $\{H_n\}$. A map $\varphi : G \to H$ is called an approximation of degree k if

(a) φ is bijective and $\varphi(G_n) = H_n$ for all n;

(b) there exists a positive integer k s.t. for any $x \in G_m$ and $y \in G_n$ we have $\varphi(xy) \equiv \varphi(x)\varphi(y) \mod H_{m+n+k}$.

The maximal k for which condition (b) holds is called the degree of φ. Thus an approximation of degree ∞ is simply an isomorphism.

Proposition 4.2. *Let $\varphi : G \to H$ be an approximation map with respect to some central filtrations $\{G_n\}$ of G and $\{H_n\}$ of H.*

1) φ induces an isomorphism φ_* of the graded Lie rings of G and H given by the formula $\varphi_*(\mathrm{LT}\, g) = \mathrm{LT}\, \varphi(g)$.

2) If $G_n = \gamma_n G$, then $H_n = \gamma_n H$.

Proof. Let $x \in G_n$ and $y \in G_m$. We have: $\varphi(xy) \equiv \varphi(x)\varphi(y) \mod H_{n+m+k}$ and $\varphi(yx) \equiv \varphi(y)\varphi(x) \mod H_{n+m+k}$, whence

$$\varphi(xy) \equiv \varphi(y)\varphi(x)(\varphi(x), \varphi(y)) \equiv \varphi(yx) \cdot (\varphi(x), \varphi(y)) \mod H_{n+m+k}.$$

On the other hand,

$$\varphi(xy) = \varphi(yx \cdot (x,y)) \equiv \varphi(yx)\varphi((x,y)) \mod H_{n+m+\min(n,m)+k}.$$

Therefore, $(\varphi(x), \varphi(y)) \equiv \varphi((x, y)) \mod H_{n+m+k}$, and the first part of the proposition easily follows.

To prove the second part we use induction on n. Suppose we have already shown that $H_n = \gamma_n H$. Let $l > n$ and $x \in H_l$. Then $x = \varphi(y)$ for some $y \in G_l = \gamma_l G$. Now y can be written in the form $\prod_i (z_i, t_i)$ where $z_i \in \gamma_{l-1} G$ and $t_i \in G$. We have:

$$\varphi(y) = \varphi(\prod (z_i, t_i)) \equiv \prod \varphi((z_i, t_i)) \equiv \prod (\varphi(z_i), \varphi(t_i)) \mod H_{l+k}.$$

Now $(\varphi(z_i), \varphi(t_i)) \in (H_{l-1}, H) \subseteq (\gamma_n H, H) = \gamma_{n+1} H$. It follows that $H_l \subseteq H_{l+k}\gamma_{n+1}H \subseteq H_{l+2k}\gamma_{n+1}H \subseteq \cdots \subseteq \gamma_{n+1}H$. On the other hand, $H_{n+1} \supseteq (H_n, H) = \gamma_{n+1}H$. The proof is completed. $\square$

Now we are ready to show that $\varphi_{s,r} : SL_2^1(\mathbb{F}_p[[t]]) \to \mathcal{Q}^1(s, r)$ is an approximation map. We will keep the notations of Proposition 4.1.

Proposition 4.3. *The map $\varphi : G \to H$ is an approximation map with respect to the filtrations $\{G_n\}$ and $\{H_n\}$, and $\deg(\varphi) \geq (q-1)/2$, where $q = p^s$.*

Proof. Let $g_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $g_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be two elements of G with $\deg g_1 = n$ and $\deg g_2 = m$. Set $h_1 = \varphi(g_1)$ and $h_2 = \varphi(g_2)$. We have

$$\varphi(g_1)\varphi(g_2) = h_1 h_2 = \left[\sqrt[r]{\frac{a_{11}^q t^r + a_{21}^q}{a_{12}^q t^r + a_{22}^q}}\right] \cdot \left[\sqrt[r]{\frac{b_{11}^q t^r + b_{21}^q}{b_{12}^q t^r + b_{22}^q}}\right] =$$
$$\left[\sqrt[r]{\frac{(\tilde{b}_{11}^q a_{11}^q + \tilde{b}_{21}^q a_{12}^q)t^r + (\tilde{b}_{11}^q a_{21}^q + \tilde{b}_{21}^q a_{22}^q)}{(\tilde{b}_{12}^q a_{11}^q + \tilde{b}_{22}^q a_{12}^q)t^r + (\tilde{b}_{12}^q a_{21}^q + \tilde{b}_{22}^q a_{22}^q)}}\right],$$

where $\tilde{b}_{ij} = b_{ij} \circ h_1(t)$, and

$$\varphi(g_1 g_2) = \left[\sqrt[r]{\frac{(a_{11}^q b_{11}^q + a_{12}^q b_{21}^q)t^r + (a_{21}^q b_{11}^q + a_{22}^q b_{21}^q)}{(a_{11}^q b_{12}^q + a_{12}^q b_{22}^q)t^r + (a_{21}^q b_{12}^q + a_{22}^q b_{22}^q)}}\right].$$

We need to prove that $\varphi(g_1 g_2) \equiv \varphi(g_1)\varphi(g_2) \mod H_{n+m+(q-1)/2}$. According to part 3) of Proposition 4.1, this is equivalent to showing that $a_{i1}\tilde{b}_{1j} + a_{i2}\tilde{b}_{2j} - a_{i1}b_{1j} - a_{i2}b_{2j} \in t^{n+m+(q-1)/2}\mathbb{F}_p[[t]]$ for $i, j = 1, 2$. Clearly, it will be sufficient to check that $\tilde{b}_{ij} - b_{ij} \in t^{n+m+(q-1)/2}\mathbb{F}_p[[t]]$ for $i, j = 1, 2$. We'll do the case $i = j = 1$; the other cases are analogous.

We know that $b_{11} = 1 + \sum_{i \geq m} \beta_i t^i$, whence $\tilde{b}_{11} = 1 + \sum_{i \geq m} \beta_i (h_1(t))^i$. Since $h_1(t) \in \mathcal{Q}_{qn-r}$, we have $h_1(t) = t(1 + \sum_{j \geq qn-r} \mu_j t^j)$. Therefore,

$$\tilde{b}_{11} = 1 + \sum_{i \geq m} \beta_i t^i (1 + \sum_{j \geq qn-r} \mu_j t^j)^i \equiv b_{11} \mod t^{m+qn-r}\mathbb{F}_p[[t]].$$

It remains to note that $m+qn-r \geq n+m+q-1-r \geq n+m+(q-1)/2$ $\square$

Remark. We have actually shown that $\lim_{s \to \infty} \deg \varphi_{s,r} = \infty$, i.e. as $s \to \infty$ the groups $\{\mathcal{Q}^1(s,r)\}$ "converge" to $SL_2^1(\mathbb{F}_p[[t]])$.

Since $G_n = \gamma_n G$, the last two propositions imply that $H_n = \gamma_n H$, and $\varphi_{s,r}$ induces an isomorphism of the graded Lie algebras of $SL_2^1(\mathbb{F}_p[[t]])$ and $\mathcal{Q}^1(s,r)$ with respect to the lower central series. Therefore the Lie algebra of $\mathcal{Q}^1(s,r)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$ as a graded Lie algebra. The fact that $\mathcal{Q}^1(s,r)$ is hereditarily just-infinite easily follows from this.

Before completing the proof of Theorem 1.1, we remark that $SL_2^1(\mathbb{F}_p[[t]])$ is not the unique linear group whose Lie algebra with respect to the lower central series is isomorphic to $\mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$ – the same is true for the group $SL_2^1(\mathbb{Z}_p)$. However, these two groups have different Lie algebras with respect to the Zassenhaus filtration. In the case of $SL_2^1(\mathbb{F}_p[[t]])$, the Zassenhaus filtration coincides with the lower central series, and the associated Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$ as a restricted Lie algebra. On the other hand, the Lie algebra of $SL_2^1(\mathbb{Z}_p)$ with respect to the Zassenhaus filtration is abelian. Theorem 1.1 asserts that the groups $\mathcal{Q}^1(s,r)$ and $SL_2^1(\mathbb{F}_p[[t]])$ cannot be distinguished in this way.

Note that since the Lie algebra $\mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$ is centerless, any two restricted structures on it must be identical. Therefore, to finish the proof of Theorem 1.1, it suffices to show that the Zassenhaus filtration of $\mathcal{Q} = \mathcal{Q}^1(s,r)$ coincides with the lower central series. The latter amounts to checking the inclusion $(\gamma_i \mathcal{Q})^p \subseteq \gamma_{pi}\mathcal{Q}$ for all i. We know that $\gamma_i \mathcal{Q} = \mathcal{Q}_{qi-r}$, where $\mathcal{Q}_j = \mathcal{Q} \cap \mathcal{N}_j$ as before. Recall that $\mathcal{Q}_{qi-r} = \langle f \rangle \mathcal{Q}_{qi}$ where f is an arbitrary element of $\mathcal{Q}_{qi-r} \backslash \mathcal{Q}_{qi}$. It will be convenient for us to take $f = f_{qi-r} = \left[t(1+t^{qi-r})^{-1/(qi-r)}\right]$ since this element has order p. Now any $g \in \mathcal{Q}_{qi-r}$ can be written in the form $g = hf^\lambda$ where $h \in \mathcal{Q}_{qi}$ and $\lambda \in \mathbb{N}$. We need to show that $g^p \in \mathcal{Q}_{pqi-r}$. It follows from Hall-Petrescu formula that $(hf^\lambda)^p = h^p f^{p\lambda} \cdot w_1 \cdot w_2$ where w_1 is a product of commutators in h and f^λ of length at least p and w_2 is a product of p^{th} powers of commutators of length at least 2.

We have $h^p \in \mathcal{Q}_{pqi}$, $(f_{qi-r})^{p\lambda} = 1$, $w_1 \in (\gamma_i\mathcal{Q}, \ldots, \gamma_i\mathcal{Q}) \subseteq \gamma_{pi}\mathcal{Q}$ since $h, f_{qi-r} \in \gamma_i\mathcal{Q}$, and $w_2 \in (\mathcal{N}_{qi-r}, \mathcal{N}_{qi-r})^p \subseteq \mathcal{N}_{2p(qi-r)}$. Since $2p(qi-r) \geq 2pqi - pq > pqi - r$, all four factors lie in $\gamma_{pi}\mathcal{Q}$. $\square$

5 Hausdorff dimension and rigid groups

We start by reviewing some basic facts about Hausdorff dimension (see [BSh] and [Sh] for more details).

Let G be a profinite group with fixed filtration $\{G_n\}$. As already stated in the introduction, the Hausdorff dimension of a subgroup H of G can be computed by the formula

$$\mathrm{Hdim}_G H = \liminf_{n\to\infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|}.$$

It is clear that the Hausdorff dimension of a subgroup is a real number between 0 and 1, and that open subgroups always have dimension 1.

If G is a pro-p group and the filtration $\{G_n\}$ is p-central, the Hausdorff dimension of a subgroup H is completely determined by its Lie algebra $L_G(H)$. In particular, the formula for Hausdorff dimension can be rewritten as follows. Given a graded Lie algebra $L = \bigoplus_{n=1}^{\infty} L_n$ and its graded subalgebra $M = \bigoplus_{n=1}^{\infty} M_n$, define the lower density of M in L to be $\mathrm{ldense}_L(M) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \dim M_k}{\sum_{k=1}^{n} \dim L_k}$. One can now check that $\mathrm{Hdim}_G(H) = \mathrm{ldense}_{L(G)} L(H)$.

The following simple property of Hausdorff dimension will be frequently used. Let $C \subseteq B \subseteq A$ be profinite groups, let $\{A_n\}$ be some filtration of A, and let $\{B_n\}$ and $\{C_n\}$ be the induced filtrations on B and C. Then

$$\mathrm{Hdim}_A C \geq \mathrm{Hdim}_B C \cdot \mathrm{Hdim}_A B \tag{5.1}$$

where dimension is computed with respect to the above filtrations. If either $\mathrm{Hdim}_A B$ or $\mathrm{Hdim}_B C$ is pure, we have equality in the above formula, and if both $\mathrm{Hdim}_A B$ or $\mathrm{Hdim}_B C$ are pure, then so is $\mathrm{Hdim}_A C$.

Determination of the set of possible Hausdorff dimensions of subgroups of a given group G, called the spectrum of G, can be considered as a first step in understanding the subgroup structure of G. The knowledge of the spectrum can also be used to prove that two given groups are not commensurable (see [BK] for examples of such applications). The problem with the above notion is its dependence on the choice of a filtration. If G is pro-p, one may elect to use some natural filtration, e.g. Zassenhaus filtration, but in that case it is hard to control how the filtration changes if we replace a group by a finite index subgroup. In this section we introduce the notion of a rigid group. Hausdorff dimension of subgroups of groups with this property will be independent of the choice of a filtration under very mild assumptions on the latter. We remark that our notion of rigidity is a variation of the concept of finite obliquity introduced in [KLP].

Definition. Let G be a profinite group. A filtration $\{\Gamma_n\}$ of G is called strongly rigid if there exists $e \in \mathbb{N}$ such that for any subgroup $N \trianglelefteq_o G$ we have $\Gamma_{n+e} \subseteq N \subseteq \Gamma_n$ for some $n \in \mathbb{N}$.

Definition. A profinite group G is called rigid if there exists $C \in \mathbb{N}$ such that for any $N_1, N_2 \trianglelefteq_o G$ either $|N_1 : N_1 \cap N_2| < C$ or $|N_2 : N_1 \cap N_2| < C$.

Definition. A profinite group G is called strongly rigid if some finite width filtration of G is rigid.

Remark. It is easy to see that a strongly rigid group is rigid and just-infinite.

The following basic properties of rigid groups will be used in this paper.

Lemma 5.1. *Let G be a profinite group and let H be an open subgroup of G. If H is rigid, then so is G.*

Proof. Let $A, B \trianglelefteq_o G$. Then $A \cap H, B \cap H \trianglelefteq_o H$. Since $|A : A \cap H|$ and $|B : B \cap H|$ are bounded by $|G : H|$, the result follows. $\square$

Lemma 5.2. *Let G be a group which has a filtration consisting of rigid subgroups. Then any group commensurable to G is rigid.*

Proof. Let H be commensurable to G. Find open subgroups $A \subseteq H$ and $B \subseteq G$ such that $A \cong B$. By assumption B contains an open rigid subgroup, whence H is rigid by the previous lemma. $\square$

Lemma 5.3. *Let G be a rigid group and let H be a subgroup of G.*

1) H has the same Hausdorff dimension with respect to all finite width filtrations of G.

2) If H has pure dimension with respect to some finite width filtration of G, then H has absolute dimension.

Proof. We'll prove the first assertion of the lemma; the proof of the second assertion is completely analogous. Let H be a subgroup of G, and let $\{A_n\}$ and $\{B_n\}$ be two finite width filtrations of G. Set

$$a(n) = |H : H \cap A_n|, \quad a_0(n) = |G : A_n|, \quad \alpha_n = \log a(n)/\log a_0(n),$$
$$b(n) = |H : H \cap B_n|, \quad b_0(n) = |G : B_n|, \quad \beta_n = \log b(n)/\log b_0(n).$$

It suffices to show that $\mathrm{Limset}\,(\{\alpha_n\}) = \mathrm{Limset}\,(\{\beta_n\})$, where $\mathrm{Limset}\,(\cdot)$ stands for the set of limit points of a sequence.

Let $c = \max_{n\in\mathbb{N}} |B_n : B_{n+1}|$. Then for any $n \in \mathbb{N}$ we can find $m = m(n)$ such that $a_0(n) \leq b_0(m) \leq c \cdot a_0(n)$. Since G is rigid, there exists a constant c' such that $|A_n : A_n \cap B_m| < c'$ and $|B_m : A_n \cap B_m| < c'$. Now,

$$a(n) \leq |H : H \cap A_n \cap B_m| \leq |H : H \cap B_m| \cdot |B_m : A_n \cap B_m| \leq c' b(m),$$

and similarly $b(m) \leq c' a(n)$.

Obtained estimates clearly imply that $\lim_{n\to\infty} |\alpha_n - \beta_{m(n)}| = 0$, whence $\operatorname{Limset}(\{\alpha_n\}) \subset \operatorname{Limset}(\{\beta_n\})$. The reverse inequality is proved in the same way. □

Proposition 5.4. *A Lie simple pro-p group is hereditarily rigid.*

Proof. Let G be a Lie simple pro-p group and let $\{G_n\}$ be a finite width filtration of G such that the associated Lie algebra $L = L(G)$ is isomorphic to $\mathfrak{g} \otimes t\mathbb{F}_p[t]$ as a graded Lie algebra, where $\mathfrak{g}$ is simple over $\mathbb{F}_p$. Denote by $\{L^n\}$ be the corresponding filtration of L. By Lemma 5.2 it'll be sufficient to show that the subgroup G_n of G is rigid for any n. Let n be fixed. Take $g \in G_k \backslash G_{k+1}$ where $k \geq n$ and let $u = \operatorname{LT} g$. Then $u = xt^k$ for some $x \in \mathfrak{g}$. Let $I = [u, L^n]$ be the ideal of L^n generated by u. Using simplicity of $\mathfrak{g}$ it's easy to show that $I \supseteq L^{k+e}$ where e depends only on n. On the other hand, $I \subseteq L_G(g^{G_n})$, and therefore $G_{k+e} \subseteq g^{G_n} \subseteq G_k$, whence G_n is rigid. □

Remark. Similarly one can show that the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ is hereditarily rigid.

6 The centralizers of elements of order p in $\mathcal{N}(\mathbb{F}_p)$

In this section we give an explicit description of the centralizers of elements of order p in the Nottingham group and use it to prove Theorems 1.5 and 1.8. We start with the following easy observation.

Lemma 6.1. *Regard $\mathcal{N} = \mathcal{N}(\mathbb{F}_p)$ as a subgroup of the group of automorphisms of $\mathbb{F}_p((t))$. Let $H \subset \mathcal{N}$ be a finite subgroup and let K be the fixed field of H. Then the normalizer of H in $\mathcal{N}$ coincides with the set* $\operatorname{Stab}_{\mathcal{N}} K = \{g \in \mathcal{N} \mid g(K) = K\}$.

Proof. First of all we remark that since $\mathcal{N}$ is a pro-p group, the conditions $g(K) = K$ and $g(K) \subseteq K$ are equivalent. By Galois theory we know that $H = \{g \in \mathcal{N} \mid g|_K = id\}$. Take any $g \in \mathcal{N}$ such that $g(K) \subseteq K$. Then for any $h \in H$ and any $a \in K$ we have $ghg^{-1}a = gg^{-1}a = a$ (since $g^{-1}a \in K$).

Thus $ghg^{-1} \in H$. In the other direction, if $ghg^{-1} = h' \in H$, then for any $a \in K$ we have $ghg^{-1}a = h'a = a$, whence $h(g^{-1}a) = g^{-1}a$. Therefore $g^{-1}(K) \subseteq K$. $\square$

Proof of Theorem 1.5:

Let f be an element of order p. By Theorem 1.4 it suffices to consider the case $f = f_n = \left[\dfrac{t}{\sqrt[n]{1-t^n}}\right]$. Let K_n be the fixed field of f_n. We claim that $K_n = \mathbb{F}_p((s_n))$, where $s_n = \dfrac{t^p}{\sqrt[n]{1-t^{n(p-1)}}}$. Indeed, we have:

$$f_n(s_n) = \frac{t^p}{\sqrt[n]{1-t^{n(p-1)}}} \circ \frac{t}{\sqrt[n]{1-t^n}} = \frac{\left(\dfrac{t}{\sqrt[n]{1-t^n}}\right)^p}{\sqrt[n]{1-\dfrac{t^{n(p-1)}}{(1-t^n)^{p-1}}}}$$

$$= \frac{t^p}{\sqrt[n]{(1-t^n)^p - t^{n(p-1)}(1-t^n)}} = \frac{t^p}{\sqrt[n]{1-t^{n(p-1)}}} = s_n.$$

Thus f_n fixes $\mathbb{F}_p((s_n))$. On the other hand, $[\mathbb{F}_p((t)) : K_n] = \operatorname{card}\langle f_n\rangle = p$, whence $K_n = \mathbb{F}_p((s_n))$.

Lemma 6.2. *Let f_n, K_n be as above. Set $A = \mathrm{Stab}_{\mathcal{N}}(K_n)$, $B = \mathrm{Cent}_{\mathcal{N}}(f_n)$ and $C = \left\{\dfrac{t}{\sqrt[n]{1+t^n\alpha}} \mid \alpha \in K_n \text{ and } t^n\alpha \in t\mathbb{F}_p[[t]]\right\}$. Then $A = B = C$.*

Proof. First of all notice that $\mathrm{Cent}_{\mathcal{N}}(f_n) = \mathrm{Norm}_{\mathcal{N}}(\langle f_n\rangle)$, since the elements f_n^i and f_n^j with $i \not\equiv j \mod p$ have distinct leading terms and thus cannot be conjugate. Therefore $A = B$ by Lemma 6.1. Now we will prove that $B = C$. Pick $g \in \mathcal{N}$ and write $g(t)$ in the form $\dfrac{t}{\sqrt[n]{1-t^n\alpha}}$ for some $\alpha \in \mathbb{F}_p((t))$. Let us compute $g \cdot f_n$ and $f_n \cdot g$. We have

$$g \cdot f_n(t) = \frac{t}{\sqrt[n]{1-t^n}} \circ \frac{t}{\sqrt[n]{1-t^n\alpha}} = \frac{\dfrac{t}{\sqrt[n]{1-t^n\alpha}}}{\sqrt[n]{1-\dfrac{t^n}{1-t^n\alpha}}} = \frac{t}{\sqrt[n]{1-t^n\alpha-t^n}},$$

$$f_n \cdot g(t) = \frac{t}{\sqrt[n]{1-t^n\alpha}} \circ \frac{t}{\sqrt[n]{1-t^n}} = \frac{\dfrac{t}{\sqrt[n]{1-t^n}}}{\sqrt[n]{1-\dfrac{t^n}{1-t^n}\tilde{\alpha}}} = \frac{t}{\sqrt[n]{1-t^n\tilde{\alpha}-t^n}},$$

where $\tilde{\alpha} = \alpha \circ \dfrac{t}{\sqrt[n]{1-t^n}}$. But $\alpha \circ \dfrac{t}{\sqrt[n]{1-t^n}} = f_n(\alpha)$. Thus g commutes with f_n if and only if $f_n(\alpha) = \alpha$, which is equivalent to saying that $\alpha \in K_n$ $\square$

Denote the centralizer of f_n by C_n. Since $K_n \cong \mathbb{F}_p((t))$, we obtain a homomorphism θ_n from C_n to $\mathcal{N}(\mathbb{F}_p)$ defined as follows: take $g \in C_n$; then

$$g(s_n) = s_n(1 + g_1 s_n + g_2 s_n^2 + \dots) \text{ for some } g_i \in \mathbb{F}_p.$$

Set

$$\theta_n(g) = \left[t(1 + g_1 t + g_2 t^2 + \dots)\right] \in \mathcal{N}(\mathbb{F}_p).$$

Clearly, the kernel of this map is the cyclic group of order p generated by f_n. To find the image of θ_n we compute $g(s_n)$ for $g \in C_n$. We know that $g(t) = \dfrac{t}{\sqrt[n]{1+t^n\alpha}}$, where $\alpha = \beta \circ s_n$ for some $\beta \in \mathbb{F}_p((t))$. Therefore we have

$$g(s_n) = \frac{t^p}{\sqrt[n]{1-t^{n(p-1)}}} \circ \frac{t}{\sqrt[n]{1+t^n\alpha}} = \frac{\dfrac{t^p}{\sqrt[n]{(1+t^n\alpha)^p}}}{\sqrt[n]{1 - \dfrac{t^{n(p-1)}}{(1+t^n\alpha)^{p-1}}}} =$$

$$\frac{t^p}{\sqrt[n]{(1+t^n\alpha)^p - t^{n(p-1)}(1+t^n\alpha)}} = \frac{t^p}{\sqrt[n]{1 + t^{np}\alpha^p - t^{n(p-1)} - t^{np}\alpha}} =$$

$$\frac{\dfrac{t^p}{\sqrt[n]{1-t^{n(p-1)}}}}{\sqrt[n]{1 + \dfrac{t^{np}}{1-t^{n(p-1)}}(\alpha^p - \alpha)}} = \frac{s_n}{\sqrt[n]{1 + s_n^n(\beta(s_n)^p - \beta(s_n))}}.$$

Therefore $\theta_n g\,(t) = \dfrac{t}{\sqrt[n]{1+t^n(\beta^p - \beta)}}$.

If $\gamma \in t\mathbb{F}_p[[t]]$, then the equation $\beta^p - \beta = \gamma$ always has the solution in $t\mathbb{F}_p[[t]]$, namely $\beta = -\gamma - \gamma^p - \gamma^{p^2} - \dots$. Finally, given $a \in t\mathbb{F}_p[[t]]$, the equation $\dfrac{t}{\sqrt[n]{1+t^n\gamma}} = t \cdot (1+a)$ has the solution

$$\gamma = \frac{(1+a)^{-n} - 1}{t^n}$$

which lies in $t\mathbb{F}_p[[t]]$ whenever $a \in t^{n+1}\mathbb{F}_p[[t]]$. Therefore the image of θ_n contains the congruence subgroup $\mathcal{N}_{n+1}$, and the proof is completed. $\square$

Proof of Theorem 1.8. We consider 3 cases:
Case 1: H is finite.

First of all we remark that the quotient of the normalizer of H by the centralizer of H acts faithfully on H, and since H is finite, the centralizer has finite index in the normalizer. Since $\mathcal{N}$ is rigid, it will be enough to prove that the normalizer of H has pure Hausdorff dimension $\dfrac{1}{|H|}$ with respect to the congruence filtration of $\mathcal{N}$. By Theorem 1.4 the statement is true when $|H| = p$, and we proceed by induction on $|H| = p^k$.

Let $z \in H$ be a central element of order p and denote its centralizer in G by C. If $F \subset \mathbb{F}_p((t))$ is the fixed field of z and $G' = \mathrm{Aut}^1(F) \cong \mathcal{N}(\mathbb{F}_p)$, we have a map $\theta : C \to G'$ as described in the proof of Theorem 1.5. Set $H' = \theta(H)$. Let $N' = N_{\theta(C)}(H')$ be the normalizer of H' in $\theta(C)$ and let $N = N_G(H) \cap C$. Note that $\mathrm{Cent}_G(H) \subseteq N \subseteq N_G(H)$, so it suffices to show that N has pure Hausdorff dimension $1/p^k$ in G.

Clearly, $N = \theta^{-1}(N')$. Now by induction $\mathrm{Hdim}_{G'}(N') = \dfrac{1}{p^{k-1}}$. It is easy to check that the pullback under θ of the congruence filtration of G' and the restriction of the congruence filtration of G to C are the same apart from a finite number of terms. Since $\mathrm{Ker}\,\theta$ is finite and $\mathrm{Im}\,\theta$ is cofinite, we conclude that $\mathrm{Hdim}_C(N) = \mathrm{Hdim}_{\theta(C)}(N') = \mathrm{Hdim}_{G'}(N') = \dfrac{1}{p^{k-1}}$. Therefore,

$$\mathrm{Hdim}_G(N) = \mathrm{Hdim}_G(C) \cdot \mathrm{Hdim}_C(N) = \frac{1}{p^{k-1}} \cdot \frac{1}{p} = \frac{1}{p^k}.$$

Case 2: H is infinite torsion.

In this case by Zelmanov's solution to the restricted Burnside problem H contains finite subgroups of arbitrarily large order, whence the centralizer of H has absolute dimension zero by the previous case.

Case 3: H is not torsion.

This is the only case where we need to use Wintenberger's theorem. Let $h \in H$ be an element of infinite order, let $C = \mathrm{Cent}_G(h) \supseteq \mathrm{Cent}_G(H)$, and let $J = \{j \in \mathbb{N} \mid C \cap G_j \neq C \cap G_{j+1}\}$, where $\{G_j\}$ is the congruence filtration of G. Suppose that J has nonzero upper density ε. Choose $n \in \mathbb{N}$ s.t. $p^n \varepsilon > 1$ and set $M = \deg(h^{p^n})$. Now let $j \in J$ with $j > M$ and choose $k \in C$ with $\deg(k) = j$. By Theorem 1.7 applied to the abelian group $\langle k, h \rangle$ we have $j \equiv M \mod p^n$, which contradicts the assumption on the density of J. Therefore the subgroup C has absolute dimension zero in G. $\square$

7 Nonlinearity criteria

In this section we establish two nonlinearity criteria and use them to prove Theorem 1.2.

Proof of Theorem 1.6.

In the course of the proof we will use some facts from the theory of linear algebraic groups for which [Ma, Chapter 1] is a good reference.

Suppose that G is just-infinite and linear over some pro-p ring. By a theorem of Jaikin-Zapirain [JZ] G is linear over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Since p-adic analytic groups are virtually torsion free, we are left with the second possibility. In this case by Pink's theorem [P] there exist subgroups $\Gamma_3 \lhd \Gamma_2 \lhd \Gamma_1 \lhd \Gamma = G$ where

Γ/Γ_1 is finite;

Γ_1/Γ_2 is abelian of finite exponent;

there exists a local field F of characteristic p, a connected semisimple adjoint algebraic group $\mathbb{A}$ over F with a universal cover $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$ and an open compact subgroup L of $\widetilde{\mathbb{A}}(F)$ such that Γ_2/Γ_3 is isomorphic to $\pi(L)$ as a topological group;

Γ_3 is solvable.

Since G is just-infinite, the subgroup Γ_3 is either trivial or has finite index in G. In the latter case G is a just-infinite solvable group, hence p-adic analytic. Thus we can assume that $\Gamma_3 = \{1\}$. Since G is finitely generated, Γ_2 has finite index in G. The kernel of the map π is finite, whence Γ_2 is isomorphic to an open subgroup of L which we call H. We consider 2 cases:

Case 1: $\widetilde{\mathbb{A}}$ is isotropic.

In this case there exists an F-monomorphism of F-groups $\mathbb{G}_{\mathrm{a},\mathbb{F}} \to \widetilde{\mathbb{A}}$ (where $\mathbb{G}_{\mathrm{a},\mathbb{F}}$ denotes the additive group of a field F). Embed $\widetilde{\mathbb{A}}$ as an F-subgroup of GL_n for some n, and let $\varphi : \mathbb{G}_{\mathrm{a},\mathbb{F}}(F) \to \widetilde{\mathbb{A}}(F)$ and $\psi : \widetilde{\mathbb{A}}(F) \to GL_n(F)$ be the corresponding maps between the groups of F-points. The congruence filtration of $GL_n(F)$ induces a finite width filtration on $H \subset \mathbb{A}(F)$. Since the subgroup $K = \varphi(\mathbb{G}_{\mathrm{a},\mathbb{F}}(F)) \cap H$ is isomorphic to $(\mathbb{F}_p[[t]], +)$, it will be sufficient to show that K has positive Hausdorff dimension in H with respect to the above filtration. The latter easily follows from the fact that $\theta = \psi \circ \varphi$ is a morphism of affine varieties, whence the entries of the matrix $\theta(x) = \psi\, \varphi(x)$ are polynomials in x.

Case 2: $\widetilde{\mathbb{A}}$ is anisotropic.

Since $\widetilde{\mathbb{A}}$ is simply connected, it is a direct product of finitely many almost simple groups $\mathbb{A}_i$. Each $\mathbb{A}_i$ is obtained by restriction of scalars from an absolutely almost simple group $\mathbb{B}_i$ defined and anisotropic over some finite extension F_i of F, and $\mathbb{A}_i(F)$ is isomorphic to $\mathbb{B}_i(F_i)$ as a topological group. By the classification of absolutely almost simple algebraic groups over local fields (see [Ti]), $\mathbb{B}_i(F_i)$ is isomorphic to the group $SL_1(D_i)$ of reduced norm 1 elements of a finite-dimensional division algebra D_i over F_i. Such an algebra has no elements of order p because $x^p = 1$ implies $(x-1)^p = 0$. Therefore the pro-p group $H \subseteq \widetilde{\mathbb{A}}(F)$ is torsion-free. □

Proof of Corollary 1.9. Let H be a group satisfying the conditions of the corollary and suppose that H is linear. According to Theorem 1.6, H has an open subgroup K which has an abelian subgroup S of positive Hausdorff dimension with respect to some (hence arbitrary by Lemma 5.3) finite width filtration of K. On the other hand, by Theorem 1.8, S has dimension zero in $\mathcal{N}(\mathbb{F}_p)$ with respect to the congruence filtration. But this contradicts formula (5.1) applied to the triple $S \subset K \subset \mathcal{N}(\mathbb{F}_p)$.

□

Proposition 7.1. *Let G be a pro-p group. Suppose that there exists a sequence $\{g_n \in G\}$ of torsion elements converging to the identity such that $\mathrm{Cent}_G(g_n)$ has infinite index in G for each n, but the group $\mathrm{Cent}_G(g_n)/\langle g_n\rangle$ is isomorphic to an open subgroup of G. Then G is not linear over any field (as an abstract group).*

Proof. Assume that G is linear and let n be the minimal positive integer such that some open subgroup H of G can be embedded into a linear algebraic group A of dimension n. We know that H contains some element of the sequence $\{g_n\}$, call it g. We have

$$\mathrm{Cent}_H(g) \subseteq \mathrm{Cent}_A(g) \quad \text{and} \quad \mathrm{Cent}_H(g)/\langle g\rangle \subseteq \mathrm{Cent}_A(g)/\langle g\rangle.$$

The group $\mathrm{Cent}_A(g)/\langle g\rangle$ is algebraic as a quotient of an algebraic subgroup by a Zariski closed normal subgroup. Moreover, the dimension of this group is less than n for otherwise the index $|A : \mathrm{Cent}_A(g)|$ would be finite, contrary to our assumptions. On the other hand, the group $\mathrm{Cent}_H(g)/\langle g\rangle$ is isomorphic to an open subgroup of G, thus we have a contradiction. □

Proof of Theorem 1.2.

Part 1: We need to check that $\mathcal{Q} = \mathcal{Q}^1(s,r)$ satisfies the conditions of Corollary 1.9, which is straightforward. Indeed, this group is not virtually torsion free, since the elements $f_n = \left[\dfrac{t}{\sqrt[n]{1-t^n}}\right]$ lie in $\mathcal{Q}$ when $n \equiv r \mod p^s$. It is also hereditarily rigid by Proposition 5.4.

Part 2: It will be more convenient for us to prove nonlinearity of the group $\mathcal{Q} = \mathcal{Q}(1, r)$, which is equivalent to the assertion of the theorem. Our goal is to show that $\mathcal{Q}$ satisfies the conditions of Proposition 7.1 where the sequence of elements of order p is chosen as above. We have seen in Section 6 that the centralizer of f_n in $\mathcal{N}(\mathbb{F}_p)$ consists of the elements $\left[\frac{t}{\sqrt[n]{1+t^n \cdot (\beta \circ s_n)}}\right]$, where $s_n = \frac{t^p}{\sqrt[n]{1-t^{n(p-1)}}}$, and the map

$$\varphi : \left[\frac{t}{\sqrt[n]{1+t^n(\beta \circ s_n)}}\right] \mapsto \left[\frac{t}{\sqrt[n]{1+t^n(\beta^p - \beta)}}\right]$$

is a homomorphism from $C = \mathrm{Cent}_{\mathcal{N}}(f_n)$ to $\mathcal{N}$ with $\operatorname{Ker}\varphi = \langle f_n \rangle$ and $\operatorname{Im}\varphi \supseteq \mathcal{N}_{n+1}$. It suffices to check that $\varphi(\mathcal{Q} \cap C)$ contains an open subgroup of $\mathcal{Q}$.

As we mentioned before, if $n \equiv r \mod p$, then any element of $\mathcal{Q}$ can be written in the form $\left[\sqrt[n]{\frac{at^n+b}{ct^n+d}}\right]$ for some $a, b, c, d \in \mathbb{F}_p((t^p))$. Take any $g \in \mathcal{N}(\mathbb{F}_p)$ and write it in the form $g = \left[\frac{t}{\sqrt[n]{1+t^n\gamma}}\right]$ for some γ. It follows from the above remark that $g \in \mathcal{Q}$ if and only if there exist $a, b, c, d \in \mathbb{F}_p((t^p))$ such that $\frac{at^n+b}{ct^n+d} = \frac{t^n}{1+t^n\gamma}$, which is equivalent to $\gamma = \frac{ct^n+d}{at^n+b} - \frac{1}{t^n}$. Assume now that $g = \left[\frac{t}{\sqrt[n]{1+t^n\gamma}}\right] \in \mathcal{Q} \cap \mathcal{N}_{n+1}$. Then γ is of the above form and $\gamma \in t\mathbb{F}_p[[t]]$. We know that $\operatorname{Im}\varphi \supseteq \mathcal{N}_{n+1}$ and the preimage of g is equal to $\left[\frac{t}{\sqrt[n]{1+t^n(\beta \circ s_n)}}\right]$, where $\beta = -\gamma - \gamma^p - \gamma^{p^2} - \dots$. We have

$$\beta \circ s_n = -\gamma \circ s_n - \gamma^p \circ s_n - \dots =$$

$$-\frac{\tilde{c}\frac{t^{np}}{1-t^{n(p-1)}} + \tilde{d}}{\tilde{a}\frac{t^{np}}{1-t^{n(p-1)}} + \tilde{b}} + e + \frac{1-t^{n(p-1)}}{t^{np}} = \frac{(\tilde{c}t^{np} + \tilde{d})t^n + \tilde{d}t^{np}}{(\tilde{a}t^{np} + \tilde{b})t^n + \tilde{b}t^{np}} + \frac{1}{t^{np}} + e - \frac{1}{t^n} =$$

$$\frac{a_1t^n + b_1}{c_1t^n + d_1} + e_1 - \frac{1}{t^n} = \frac{(a_1 + c_1e_1)t^n + (b_1 + d_1e_1)}{c_1t^n + d_1} - \frac{1}{t^n},$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, e, a_1, b_1, c_1, d_1, e_1 \in \mathbb{F}_p((t^p))$. Thus $\beta \circ s_n$ has the desired form, whence $\varphi^{-1}(g) \in \mathcal{Q}$. $\square$

8 Natural subgroups of the groups $\{\mathcal{Q}(s,r)\}$

In this section we discuss the subgroup structure of the groups $\{\mathcal{Q}(s,r)\}$. The following subgroups are of particular interest.

1. The upper triangular subgroup
$\mathcal{B}^+(s,r) = \{t \mapsto \sqrt[r]{at^r + b}$ for some $a, b \in \mathbb{F}_p((t^{p^s}))\} \cap \mathcal{N}(\mathbb{F}_p)$.

2. The lower triangular subgroup
$\mathcal{B}^-(s,r) = \{t \mapsto \dfrac{t}{\sqrt[r]{at^r + b}}$ for some $a, b \in \mathbb{F}_p((t^{p^s}))\} \cap \mathcal{N}(\mathbb{F}_p)$.

3. The strictly upper triangular subgroup
$\mathcal{U}^+(s,r) = \{t \mapsto \sqrt[r]{t^r + b}$ for some $b \in \mathbb{F}_p((t^{p^s}))\} \cap \mathcal{N}(\mathbb{F}_p)$.

4. The strictly lower triangular subgroup
$\mathcal{U}^-(s,r) = \{t \mapsto \dfrac{t}{\sqrt[r]{1 + t^r b}}$ for some $b \in \mathbb{F}_p((t^{p^s}))\} \cap \mathcal{N}(\mathbb{F}_p)$.

5. The diagonal subgroup $\mathcal{T}(s) = \{t \mapsto t \cdot a \mid a \in \mathbb{F}_p((t^{p^s}))\} \cap \mathcal{N}(\mathbb{F}_p)$.

Remarks.
1) Some of the above groups have been studied before. The groups $\{B^+(s,1)\}$ were shown to be hereditarily just-infinite of finite width in [BK]. The groups $\{T(s)\}$ were studied by Fesenko [F] in connection with certain number-theoretic questions. Fesenko proved that these groups are hereditarily just-infinite; it is believed that they also have finite width.

2) Let s be fixed. The definitions of all the groups above make sense as long as $p \nmid r$, and for any such r we have $\mathcal{U}^\pm(s,r) \subset \mathcal{B}^\pm(s,r) \subset \mathcal{Q}(s,r)$. As we remarked before, $\mathcal{Q}(s,r) = \mathcal{Q}(s,r')$ if $r' \equiv \pm r \mod p^s$. The situation with the groups $\{\mathcal{B}^\pm(s,r)\}$ is similar: $\mathcal{B}^\pm(s,r) = \mathcal{B}^\pm(s,r')$ if $r' \equiv r \mod p^s$, and $\mathcal{B}^+(s,r) = \mathcal{B}^-(s,r')$ if $r' \equiv -r \mod p^s$. On the other hand, the groups $\{\mathcal{U}^\pm(s,r)\}$ are all distinct; moreover, $\mathcal{U}^+(s,r) \cap \mathcal{U}^-(s',r') = \{1\}$ if $r \neq r'$.

The groups $\{\mathcal{B}^\pm(s,r)\}$ and $\{\mathcal{U}^\pm(s,r)\}$ share certain properties of their Lie group counterparts:

$$\begin{aligned}\mathcal{Q}(s,r) \quad = \quad \mathcal{B}^+(s,r)\mathcal{U}^-(s,r) \quad &= \quad \mathcal{U}^-(s,r)\mathcal{B}^+(s,r) \quad = \\ \mathcal{U}^+(s,r)\mathcal{B}^-(s,r) \quad &= \quad \mathcal{B}^-(s,r)\mathcal{U}^+(s,r),\end{aligned}$$

$$\mathcal{B}^+(s,r) \cap \mathcal{U}^-(s,r) = \mathcal{B}^-(s,r) \cap \mathcal{U}^+(s,r) = \{1\}, \quad \mathcal{B}^+(s,r) \cap \mathcal{B}^-(s,r) = \mathcal{T}(s).$$

It turns out that the analogy with matrix groups goes further, and the subgroups $\{\mathcal{B}^\pm(s,r)\}$ admit an abstract characterization as follows.

Proposition 8.1. *Let K be a non-open subgroup of $\mathcal{Q} = \mathcal{Q}(s,r)$. Then* $\operatorname{Hdim}_{\mathcal{Q}}(K) \leq 2/3$ *and the equality holds if and only of K is conjugate in $\mathcal{Q}$ to an open subgroup of $\mathcal{B}^+(s,r)$ or $\mathcal{B}^-(s,r)$.*

Let us say that $B \subset \mathcal{Q} = \mathcal{Q}(s,r)$ is a Borel subgroup if B is conjugate to $\mathcal{B}^+(s,r)$ or $\mathcal{B}^-(s,r)$. According to the above proposition, B is a Borel subgroup if and only if $\operatorname{Hdim}_{\mathcal{Q}}(B) = 2/3$ and B is maximal with this property. This characterization will play a crucial role in the proof of Theorem 1.3. The proof of Proposition 8.1 will be postponed until the next section.

Each of the groups $\mathcal{Q}(s,r)$ has another interesting family of subgroups $\{\mathcal{Q}(s,r;n)\}_{n=1}^{\infty}$ defined as follows. If $n = p^a k$ where $p \nmid k$, set $\mathcal{Q}(s,r;n) = \mathcal{Q}(s+a,r) \cap \mathcal{A}(k)$, where $\mathcal{A}(k) = \{\big[t(1 + a_k t^k + a_{2k} t^{2k} + \dots)\big]\}$. These subgroups seem to be the analogues of the subgroups of $\{SL_2(\mathbb{F}_p[[t^n]])\}_{n=1}^{\infty}$ of $SL_2(\mathbb{F}_p[[t]])$. It turns out that each of the groups $\{\mathcal{Q}(s,r;n)\}$ is commensurable to one of the groups considered before. Indeed, given s, r, k with $p \nmid r, k$, we can find u such that $uk \equiv r \mod p^s$. Let $\varphi_k : \mathcal{N}(\mathbb{F}_p) \to \mathcal{N}(\mathbb{F}_p)$ be a monomorphism defined by $[\alpha] \mapsto \left[\sqrt[k]{\alpha \circ t^k}\right]$. It's easy to see that φ_k maps $\mathcal{Q}(s,u)$ onto an open subgroup of $\mathcal{Q}(s,r) \cap \mathcal{A}(k)$.

Remark. Similarly one can show that the groups $\mathcal{Q}(s,r)$ and $\mathcal{Q}(s,r')$ can be embedded into each other as subgroups of positive Hausdorff dimension.

Now let us discuss the Hausdorff spectra of the groups $\{\mathcal{Q}(s,r)\}$ and their connection with the spectra of $\mathcal{N}(\mathbb{F}_p)$ and $SL_2^1(\mathbb{F}_p[[t]])$. It follows from the results of [BSh] that if G is any $\mathfrak{sl}_2(\mathbb{F}_p)$-group, then $\operatorname{Spec} G \subseteq [0,2/3] \cup \{1\}$. In the case $G = SL_2^1(\mathbb{F}_p[[t]])$ the whole interval $[0,2/3]$ lies in the spectrum (see [BSh]). The proof is based on the following idea. Let B and U be the subgroups of the upper triangular and the strictly upper triangular matrices in G respectively. Then $\operatorname{Hdim}_G B = 2/3$ and $\operatorname{Hdim}_G U = 1/3$, and we have $U \cong \mathbb{F}_p[[t]]$ and $B/U \cong \mathbb{F}_p[[t]]^*$. Now the groups $\mathbb{F}_p[[t]]$ and $\mathbb{F}_p[[t]]^*$ have plenty of subgroups of any given size, and the inclusion $\operatorname{Spec} G \supset [0,2/3]$ easily follows. In the case of the groups $\{\mathcal{Q}(s,r)\}$ we can say the following.

Proposition 8.2. *The following hold:*

a) $\bigcup_{n=1}^{\infty}\{1/n\} \cup \bigcup_{n=1}^{\infty}\{2/3n\} \subseteq \operatorname{Spec}\mathcal{Q}(s,r) \subseteq [0,2/3] \cup \{1\}$;

b) $\operatorname{Spec}\mathcal{Q}(1,1) \supseteq [0,1/3]$.

For the proof of part a) one just needs to check that $\operatorname{Hdim}_{\mathcal{Q}(s,r)}\mathcal{Q}(s,r;n) = 1/n$ and $\operatorname{Hdim}_{\mathcal{Q}(s,r)}\mathcal{B}(s,r;n) = 2/3n$ where $\mathcal{B}(s,r;n)$ is a Borel subgroup of $\mathcal{Q}(s,r;n)$. Part b) follows from [BK, Lemma 4.1]. The situation here is similar to the case of SL_2, as all the subgroups contributing to the interval $[0,1/3]$ can be chosen inside the subgroup $\mathcal{U}^+(1,1)$. On the other hand, it is not clear how many points from $(1/3, 2/3)$ lie in the spectrum of $\mathcal{Q}(s,r)$. The reason is that unlike the case of SL_2, the group $\mathcal{B}^+(s,r)$ is very far from being solvable.

Finally, using formula (5.1) and the fact that $\mathcal{Q}(s,r)$ has absolute Hausdorff dimension $3/p^s$ in $\mathcal{N}$, we obtain the following relation between the spectra of $\mathcal{N}$ and $\mathcal{Q}(s,r)$:

$$\mathrm{Spec}\mathcal{N} \supseteq \bigcup_{s=1}^{\infty} \bigcup_{r=1}^{(p^s-1)/2} \frac{3}{p^s}\mathrm{Spec}\mathcal{Q}(s,r).$$

Thus we conclude that $\mathrm{Spec}\mathcal{N} \supseteq \{3/(pk)\}_{k=1}^{\infty}$. Combining this with the results of [BK], we get the following information about $\mathrm{Spec}\mathcal{N}$ for $p > 2$:

$$[0, \tfrac{1}{p}] \cup \{\tfrac{3}{2p}\} \cup \{\tfrac{3}{p}\} \cup \{\tfrac{1}{n}\}_{n=1}^{\infty} \cup \{\tfrac{1}{p} + \tfrac{1}{p^s}\}_{s=1}^{\infty} \subseteq \mathrm{Spec}\mathcal{N} \subseteq [0, \tfrac{2}{p}] \cup \{\tfrac{3}{p}\} \cup \{\tfrac{1}{n}\}_{n=1}^{\infty}.$$

Remark. The fact that $\mathrm{Spec}\mathcal{N} \cap (2/p, 3/p) = \emptyset$ was never stated before, but it is actually an easy consequence of the proof of [BSh, Theorem 1.6].

As one can see, the only unknown part of $\mathrm{Spec}\mathcal{N}$ lies in the open interval $(1/p, 2/p)$. Our investigation yielded two previously unknown points in the spectrum: $\{3/p\}$ and $\{3/2p\}$.

9 Classification up to commensurability

The goal of this section is to obtain a partial classification up to commensurability of the groups in the families $\mathcal{B} = \{\mathcal{B}^+(s,r)\}$, $\mathcal{Q} = \{\mathcal{Q}(s,r)\}$ and $\mathcal{T} = \{\mathcal{T}(s)\}$.

Theorem 9.1. *a) If two groups belonging to the families $\mathcal{B}, \mathcal{Q}$ or $\mathcal{T}$ are commensurable, they belong to the same family.*
b) The groups $\{\mathcal{B}^+(s,1)\}_{s=1}^{\infty}$ are pairwise non-commensurable.
c) The groups $\{\mathcal{Q}(s,1)\}_{s=1}^{\infty}$ are pairwise non-commensurable.
d) None of the groups in these families is commensurable to $\mathcal{N}$ if $p \neq 3$.

Remark. When $p = 3$, the group $\mathcal{Q}(1,1)$ coincides with the Nottingham group $\mathcal{N}$.

Part b) of this theorem is proved in [BK]. The proof is based on the computation of certain commensurability invariants ρ^+ and ρ^- which are defined as follows. Let G be a pro-p group such that $G^{(n)}$ is open in G for all n (where $G^{(n)}$ is the n^{th} term of the derived series of G). Define $\rho^+(G) = \limsup\limits_{n\to\infty} \sqrt[n]{\log|G : G^{(n)}|}$ and $\rho^-(G) = \liminf\limits_{n\to\infty} \sqrt[n]{\log|G : G^{(n)}|}$. We will make use of these invariants as well to prove the other assertions of the theorem.

Proposition 9.2. *The following hold:*
1) $\rho^+(\mathcal{Q}(s,r)) = \rho^-(\mathcal{Q}(s,r)) = 2$ *for all* r, s.
2) $\rho^-(\mathcal{B}^+(s+1,r)) > \rho^+(\mathcal{B}^+(s,1)) > 2$ *for all* r, s.

Proof. Part 1) It is obvious that $\rho^{\pm}(\mathcal{Q}(s,r)) \geq 2$ since $G^{(n)} \subseteq \gamma_{2^n}(G)$. On the other hand, if G is a pro-p group and L is its graded Lie algebra with respect to some filtration, then $L_G(G^{(n)}) \supseteq L^{(n)}$. Thus it suffices to show that $\limsup\limits_{n\to\infty} \sqrt[n]{\log|L : L^{(n)}|} = 2$. Since ρ^+ is a commensurability invariant we can assume (by Theorem 1.1) that $L = \mathfrak{sl}_2(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$. But then clearly $\log_p|L : L^{(n)}| = 3 \cdot (2^n - 1)$, and the result follows.
Part 2) We'll need the following lemma:

Lemma 9.3. *Let* K *be a subgroup of* $\mathcal{N}$. *Set* $b_n(K) = \max\{m \mid K^{(n)} \subseteq \mathcal{N}_m\}$ *and define* $\tilde{\rho}(K) := \liminf\limits_{n\to\infty} \sqrt[n]{b_n(K)}$. *Then:*
1) If $K \subseteq L$, *then* $\tilde{\rho}(K) \geq \tilde{\rho}(L)$.
2) If $\operatorname{Hdim}_{\mathcal{N}}(K) > 0$, *then* $\tilde{\rho}(K) \leq \rho^-(K)$.

Remark. The value of $\tilde{\rho}$ depends on the embedding of K in $\mathcal{N}$.

Proof. Part 1) is obvious.
Part 2): Set $\alpha(n) = \dfrac{\log|K : K \cap \mathcal{N}_n|}{\log|\mathcal{N} : \mathcal{N}_n|}$. We have:

$$\log|K : K^{(n)}| \geq \log|K : K \cap \mathcal{N}_{b_n}| = \frac{\log|K : K \cap \mathcal{N}_{b_n}|}{\log|\mathcal{N} : \mathcal{N}_{b_n}|} \log|\mathcal{N} : \mathcal{N}_{b_n}| = \alpha(b_n)(b_n - 1),$$

whence $\frac{1}{n}\log(\log|K : K^{(n)}|) \geq \frac{1}{n}(\log\alpha(b_n) + \log(b_n - 1))$. Now, $\liminf\limits_{n\to\infty} \alpha(b_n) \geq \operatorname{Hdim}_{\mathcal{N}} K > 0$, $\liminf\limits_{n\to\infty} \frac{1}{n}\log(\log|K : K^{(n)}|) = \log\rho^-(K)$, and $\liminf\limits_{n\to\infty} \frac{1}{n}\log(b_n - 1) = \log\tilde{\rho}(K)$, and the desired inequality follows. □

Now we can finish the proof of the proposition. Pick $u \in \mathbb{N}$ such that $ru \equiv 1 \mod p^s$. Let $\varphi_u : \mathcal{N}(\mathbb{F}_p) \to \mathcal{N}(\mathbb{F}_p)$ be a monomorphism defined by $[\alpha] \mapsto \left[\sqrt[u]{\alpha \circ t^u}\right]$. It's easy to see that φ_u maps $\mathcal{B}^+(s,r)$ onto an open subgroup of $\mathcal{B}^+(s,1) \cap \mathcal{A}(u)$. Since $\mathcal{B}^+(s,1) \cap \mathcal{A}(u)$ has positive Hausdorff dimension in $\mathcal{N}$, Lemma 9.3 yields

$$\rho^-(\mathcal{B}^+(s,r)) = \rho^-(\mathcal{B}^+(s,1) \cap \mathcal{A}(u)) \geq \tilde{\rho}(\mathcal{B}^+(s,1) \cap \mathcal{A}(u)) \geq \tilde{\rho}(\mathcal{B}^+(s,1)).$$

On the other hand, from the proofs of [BK, Proposition 8.5] and [BK, Theorem 1.6] it follows that $\tilde{\rho}(\mathcal{B}^+(s+1,1)) > \rho^+(\mathcal{B}^+(s,1)) > 2$. The proof is completed. □

Proof of Theorem 9.1
The groups in the family $\mathcal{T}$ are clearly distinct from the others: they are torsion free, while the groups in the families $\mathcal{B}$ and $\mathcal{Q}$ are not virtually torsion free. The families $\mathcal{B}$ and $\mathcal{Q}$ can be separated via the invariant ρ^+ by Proposition 9.2. This finishes the proof of the first part.
Part 3) According to Proposition 8.1, $\mathcal{Q}(s,1)$ has at most 2 commensurability classes of subgroups of Hausdorff dimension 2/3, and these classes are represented by the subgroups $\mathcal{B}^+(s,1)$ and $\mathcal{B}^-(s,1) = \mathcal{B}^+(s,p^s-1)$. If $\mathcal{Q}(s,1)$ and $\mathcal{Q}(s',1)$ with $s < s'$ were commensurable to each other, then $\mathcal{B}^+(s,1)$ would be commensurable to $\mathcal{B}^+(s',1)$ or $\mathcal{B}^+(s',p^{s'}-1)$, which is clearly impossible by Proposition 9.2.
Part 4) The groups $\{\mathcal{B}^+(s,r)\}$ are not commensurable to $\mathcal{N}$ since $\rho^+(\mathcal{N})=2$. Finally, to show that $\mathcal{Q}(s,r)$ is not commensurable to $\mathcal{N}$, it suffices to note that $\mathcal{N}$ has no subgroup of Hausdorff dimension 2/3: it is proved in [BSh] that the Hausdorff dimension of a non-open subgroup of $\mathcal{N}$ can't exceed 1/2 for $p > 5$ and 3/5 for $p = 5$. Since $\mathcal{N}$ is hereditarily rigid, we are done. □

The remainder of this section is devoted to the proof of Proposition 8.1. We start with some rather general considerations.

Recall that $\{\mathcal{N}_n\}$ denotes the congruence filtration of $\mathcal{N}$. For the rest of this section we fix the following notation: if G is a subgroup of $\mathcal{N}$, then $L(G) = L_{\mathcal{N}}(G)$ will denote the graded Lie algebra of G with respect to the induced filtration $\{G \cap \mathcal{N}_n\}$. For $I \subseteq \mathbb{N}$, define $L_I := \bigoplus_{i\in I} \mathbb{F}_p e_i \subseteq L(\mathcal{N})$.

Recall that $r \in \mathcal{N}$ has degree n if $r \in \mathcal{N}_n \backslash \mathcal{N}_{n+1}$. Given an element r of degree n and a subgroup H of $\mathcal{N}$, the largest integer m such that $r \in \mathcal{N}_{n+m}H$ will be called the depth of r with respect to H and denoted by $\operatorname{dep}(r,H)$. It's clear that $\operatorname{dep}(r,H) = \infty$ if and only if $r \in H$ and $\operatorname{dep}(r,H) = 0$ if and only if $\operatorname{LT}(r) \notin L_{\mathcal{N}}(H)$.

Any element $r \in \mathcal{N}$ can be written in the form $r = h \cdot s$ where $h \in H$ and $\deg(s) = \deg(r) + \operatorname{dep}(r,H)$. Such a factorization will be referred to as a standard decomposition of r with respect to H. While it's not unique, the leading term of s is independent of the choice of decomposition (here we use the fact that all the factors $\mathcal{N}_n/\mathcal{N}_{n+1}$ are cyclic).

Now let $H \subseteq G$ be fixed subgroups of $\mathcal{N}$. We have $L(H) = L_I$ and $L(G) = L_{I_0}$ for some $I \subseteq I_0 \subseteq \mathbb{N}$. Let R be another subgroup of G such that $L(R) = L(H)$ and set $m = m(R, H) = \min_{r \in R} \operatorname{dep}(r, H)$.

For each $i \in I_0$ we choose an element of G whose leading term is equal to e_i. The particular choice of such element is not important, and by abuse of notation this element will also be denoted by e_i. Given $r \in R$, let hs be its standard decomposition. Let $n = \deg(r) = \deg(h)$. Write $h = e_n^{\alpha} \cdot h'$, $s = e_{n+m}^{\beta} \cdot s'$ with $\deg(h') > n$ and $\deg(s') > n + m$. Numbers α and β are well defined modulo p. We claim that their ratio $\dfrac{\beta}{\alpha}$ depends only on $n = \deg(r)$.

Indeed, let $r_1 = h_1 s_1$ and $r_2 = h_2 s_2$ be two elements of the same degree. Set $\alpha_1 = \alpha(r_1)$, $\alpha_2 = \alpha(r_2)$, $\beta_1 = \beta(r_1)$, $\beta_2 = \beta(r_2)$ and consider the element $r = r_1^{\alpha_2} \cdot r_2^{-\alpha_1}$. We have

$$r = r_1^{\alpha_2} \cdot r_2^{-\alpha_1} = (h_1 s_1)^{\alpha_2} \cdot (h_2 s_2)^{-\alpha_1} \equiv h_1^{\alpha_2} \cdot h_2^{-\alpha_1} \cdot s_1^{\alpha_2} \cdot s_2^{-\alpha_1} \mod \mathcal{N}_{n+m+1}.$$

Note that $h_1^{\alpha_2} \cdot h_2^{-\alpha_1} \in \mathcal{N}_{n+1}$ and $s_1^{\alpha_2} \cdot s_2^{-\alpha_1} \equiv e_{n+m}^{\alpha_2\beta_1 - \alpha_1\beta_2} \mod \mathcal{N}_{n+m+1}$. Since $\operatorname{dep}(r, H) \geq m$, it follows that $\alpha_2\beta_1 - \alpha_1\beta_2 = 0$ (here we use the fact that $\operatorname{LT}(s_1), \operatorname{LT}(s_2) \notin L_I$ by the definition of a standard decomposition, whence $e_{n+m} \notin L_I$ unless $\beta_1 = \beta_2 = 0$, and in the latter case there is nothing to prove).

Let us denote the ratio $\dfrac{\beta}{\alpha}$ by $\lambda(n)$. We can view λ as a function from I to $\mathbb{F}_p$. Notice that λ is not identically zero unless $R = H$. Here are the main properties of this function.

Lemma 9.4. *Let $i, j \in I$. The following hold:*
a) if $i + m \in I$ or $i + m \notin I_0$, then $\lambda(i) = 0$;
b) if $i + j + m \notin I$, then

$$(j - i)(\lambda(i + j) - \lambda(i) - \lambda(j)) = m(\lambda(j) - \lambda(i)).$$

Remark. If $i + j \notin I$, we define $\lambda(i + j)$ to be any number. The latter can only happen when $[e_i, e_j] = 0$, i.e. when $j - i \equiv_p 0$, whence our choice doesn't affect the value of the left hand side.

Proof. Part a) is obvious. To prove part b) choose $r_1, r_2 \in R$ with $\deg(r_1) = i$, $\deg(r_2) = j$ and assume for convenience that $r_1 \equiv e_i \mod \mathcal{N}_{i+1}$ and $r_2 \equiv e_j \mod \mathcal{N}_{j+1}$. Let $r_1 = h_1 s_1$ and $r_2 = h_2 s_2$ be standard decompositions of these elements. Then we have $s_1 \equiv e_{i+m}^{\lambda(i)} \mod \mathcal{N}_{i+m+1}$ and $s_2 \equiv e_{j+m}^{\lambda(j)} \mod \mathcal{N}_{j+m+1}$.

Consider the element (r_1, r_2). We have $(r_1, r_2) \equiv (h_1, h_2) \cdot (s_1, h_2) \cdot (h_1, s_2)$ mod $\mathcal{N}_{i+j+m+1}$; moreover, $(h_1, h_2) \equiv e_{i+j}^{j-i} \mod \mathcal{N}_{i+j+1} \cap H$ and

$$(h_1, s_2) \cdot (s_1, h_2) \equiv e_{i+j+m}^{\lambda(j)(j+m-i)+\lambda(i)(j-m-i)} \mod \mathcal{N}_{i+j+m+1}.$$

Since $\operatorname{dep}((r_1, r_2), H) \geq m$, part b) of the lemma follows from these formulas.

$\square$

Proof of Proposition 8.1.

Recall that the Hausdorff dimension of K in $\mathcal{Q} = \mathcal{Q}(s, r)$ is equal to the density of $L(K)$ in $L(\mathcal{Q})$. Since $L(\mathcal{Q})$ is commensurable to $\mathfrak{sl}_2(t\mathbb{F}_p[t])$, it follows from [BShZ] that the density of $L(K)$ is at most $2/3$, and the equality is obtained if and only if $L(K)$ is cofinite in $\mathfrak{b}^-(s, r) = \bigoplus_{n \equiv 0, r \mod q} \mathbb{F}_p e_n$ or in $\mathfrak{b}^+(s, r) = \bigoplus_{n \equiv 0, -r \mod q} \mathbb{F}_p e_n$, where $q = p^s$. Both cases are analogous and we'll treat the second one. Clearly there exists $M \in \mathbb{N}$ such that $L(K \cap \mathcal{N}_M) = L_I$ where $I = \{i \in \mathbb{N} \mid i \equiv_q 0, -r \text{ and } i \geq M\}$. Our goal is to prove that $K \cap \mathcal{N}_M$ is conjugate to $\mathcal{B}^+(s, r) \cap \mathcal{N}_M$. The following proposition which can be proved by adapting the argument of [Pa, Lemma 4] enables us to "guess" the conjugating element right away.

Proposition 9.5. *Let $h \in \mathcal{Q}(s, r)$ with $p^s \mid \deg(h)$. Then there exists $g \in Q(s, r)$ such that $ghg^{-1} \in \mathcal{T}(s)$.*

Now choose $g \in Q(s, r)$ such that $g(K \cap \mathcal{N}_M)g^{-1} \cap \mathcal{T}(s) \neq \{1\}$. We claim that $g(K \cap \mathcal{N}_M)g^{-1} = \mathcal{B}^+(s, r) \cap \mathcal{N}_M$. Let $G = \mathcal{Q}(s, r)$, $R = g(K \cap \mathcal{N}_M)g^{-1}$, $H = \mathcal{B}^+(s, r) \cap \mathcal{N}_M$ and set $m = m(R, H)$. We're going to apply Lemma 9.4 to the function λ associated to this triple to prove λ is identically zero whence $H = R$. We have $I = \{i \in \mathbb{N} \mid i \equiv_q 0, -r \text{ and } i \geq M\}$ and $I_0 = \{i \in \mathbb{N} \mid i \equiv_q 0, \pm r\}$. We consider three cases:

Case 1: $m \not\equiv_q r, 2r$.

In this case λ is identically zero by Lemma 9.4 a).

Case 2: $m \equiv_q r$.

If $i \equiv_q -r$ then $i + m \in I$, whence $\lambda(i) = 0$. Now take $i \equiv_q j \equiv_q 0$. Since $i + j + m \notin I$, Lemma 9.4 b) yields $m(\lambda(i) - \lambda(j)) = 0$. But we know that $\lambda(i_0) = 0$ for some $i_0 \equiv_q 0$ since $R \cap \mathcal{T}(s) \neq \{1\}$. Therefore λ is identically zero.

Case 3: $m \equiv_q 2r$.

If $i \equiv_q 0$ then $i + m \notin I_0$, whence $\lambda(i) = 0$. Now take $i \equiv_q 0$ and $j \equiv_q -r$. Once again part b) of Lemma 9.4 can be applied, and we have $-r(\lambda(i+j) - \lambda(j)) = 2r\lambda(j)$, whence $-\lambda(i+j) = \lambda(j)$. The same argument applied to the pairs $(i, i+j)$ and $(2i, j)$ yields $-\lambda(2i+j) = \lambda(i+j)$ and $-\lambda(2i+j) = \lambda(j)$. It follows from these three equalities that $\lambda(j) = 0$.

To finish the proof of the proposition we note that gKg^{-1} normalizes $g(K \cap \mathcal{N}_M)g^{-1}$. On the other hand, the normalizer of $\mathcal{B}^+(s,r) \cap \mathcal{N}_M$ in $\mathcal{Q}(s,r)$ coincides with $\mathcal{B}^+(s,r)$ since the latter group is maximal non-open in $\mathcal{Q}(s,r)$. Therefore $gKg^{-1} \subseteq \mathcal{B}^+(s,r)$. $\square$

Remark. Using the same method one can prove that any subgroup of $\mathcal{N}(\mathbb{F}_p)$ of Hausdorff dimension $3/p$ is conjugate to an open subgroup of $\mathcal{Q}(1,r)$ for some r.

10 Open questions

1. In this paper we answered the questions from [Ba] for $n = 2$. Do there exist non-linear pro-p groups with Lie algebras isomorphic to $\mathfrak{sl}_n(\mathbb{F}_p) \otimes t\mathbb{F}_p[t]$ for $n > 2$? While the answer to this question may very well be positive, it seems that potential examples must be of completely different nature from the groups $\{\mathcal{Q}^1(s,r)\}$. Still one can find natural multi-dimensional generalizations of the groups $\{\mathcal{Q}^1(s,r)\}$ among subgroups of the Cartan-type groups $W_n = \mathrm{Aut}^{\,1}\mathbb{F}_p[[x_1, \ldots x_n]]$. For example, when $s = r = 1$, the group $\mathcal{Q}^1(1,1)$ is the first one in the family $\{\mathcal{Q}^1_n(1,1)\}_{n=1}^{\infty}$, where $\mathcal{Q}^1_n(1,1)$ is the subgroup of W_n consisting of the elements

$$f : x_i \mapsto \frac{\sum_{j=1}^{n} a^p_{i,j} x_j + a^p_{i,n+1}}{\sum_{j=1}^{n} a^p_{n+1,j} x_j + a^p_{n+1,n+1}} \text{ for } 1 \le i \le n$$
$$\text{s.t. } a_{ij} - \delta_{ij} \in \mathfrak{m}_n, \text{ where } \mathfrak{m}_n \text{ is the maximal ideal of } \mathbb{F}_p[[x_1, \ldots x_n]].$$

A related family consists of the groups $\tilde{\mathcal{Q}}^1_n(1,1) = \{f \in W_{n+1} \mid f(x_i) = \sum_{j=1}^{n+1} a^p_{i,j} x_j \text{ for } 1 \le i \le n+1 \text{ s.t. } a_{ij} - \delta_{ij} \in \mathfrak{m}_{n+1} \text{ and } \det((a_{ij})) = 1\}$.

The Lie algebras of the groups $\mathcal{Q}^1_n(1,1)$ and $\tilde{\mathcal{Q}}^1_n(1,1)$ with respect to their lower central series are isomorphic to $\mathfrak{sl}_{n+1}(\mathbb{F}_p) \otimes \mathfrak{m}_n$ and $\mathfrak{sl}_{n+1}(\mathbb{F}_p) \otimes \mathfrak{m}_{n+1}$ respectively and thus don't have finite width. These groups appear to be just-infinite unlike the linear groups $SL_n\mathbb{F}_p[[x_1, \ldots x_k]]$ with $k > 1$.

2. In the course of the proof of Theorem 1.1 we've shown that the groups $\{\mathcal{Q}^1(s,r)\}$ in some sense converge to $SL^1_2(\mathbb{F}_p[[t]])$ as $s \to \infty$. Can this fact be used to find more similarities between $SL^1_2(\mathbb{F}_p[[t]])$ and $\mathcal{Q}^1(s,r)$ for s large enough? For example, is the group $\mathcal{Q}^1(s,r)$ finitely presented? Finite presentability of $SL^1_n(\mathbb{F}_p[[t]])$ was established in [LSh], but the proof

presented there is not Lie-theoretic and doesn't extend to non-linear groups. If the answer to the above question is positive for s large enough, it is likely to imply that the Nottingham group is finitely presented.

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