GOLOD-SHAFAREVICH GROUPS WITH PROPERTY (T) AND KAC-MOODY GROUPS

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Abstract. We construct Golod-Shafarevich groups with property (T) and thus provide counterexamples to a conjecture stated in a recent paper of Zelmanov [Ze2]. Explicit examples of such groups are given by lattices in certain topological Kac-Moody groups over finite fields. We provide several applications of this result including examples of residually finite torsion non-amenable groups.

1. Introduction

In 1964, Golod and Shafarevich [GS] found a sufficient condition for a group given by generators and relators to be infinite. A slightly improved version of their result, due to Vinberg and Gaschütz, asserts the following:

Theorem 1.1. *Let p be a prime. Let Γ be a finitely generated group whose pro-p completion $\Gamma_{\widehat{p}}$ has a presentation with d generators and r relators such that $r < d^2/4$ and d is the minimal number of generators for $\Gamma_{\widehat{p}}$. Then $\Gamma_{\widehat{p}}$ is infinite (and so is Γ).*

The inequality $r < d^2/4$ is a special case of the so-called Golod-Shafarevich condition, which guarantees that a pro-p group given by generators and relators is infinite.

Definition. Let G be a pro-p group. Let $\langle X|R\rangle$ be a pro-p presentation of G, and let r_i be the number of defining relators of degree i with respect to the Zassenhaus filtration.

a) We say that the above presentation satisfies the *Golod-Shafarevich condition* if there exists $0 < t < 1$ such that $1 - dt + \sum_{i=1}^{\infty} r_i t^i < 0$.
b) We say that G is a *Golod-Shafarevich group* if it has a presentation satisfying the Golod-Shafarevich condition.

Definition. A discrete group Γ is Golod-Shafarevich (with respect to p)[1] if its pro-p completion $\Gamma_{\widehat{p}}$ is Golod-Shafarevich.

Remark: If a discrete group Γ is given by a presentation $\langle X|R\rangle$, then $\Gamma_{\widehat{p}}$ is given by the same presentation $\langle X|R\rangle$ in the category of pro-p groups.

It is easy to show (see subsection 2.3) that a discrete group Γ is Golod-Shafarevich provided $d(\Gamma_{\widehat{p}}) > 1$ and Γ has a presentation $\langle X|R\rangle$ with $|R| < |X| - d(\Gamma_{\widehat{p}}) + d(\Gamma_{\widehat{p}})^2/4$ (here $d(\Gamma_{\widehat{p}})$ is the minimal number of generators

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[1] The reference to p will be omitted when clear from the context.

for $\Gamma_{\widehat{p}}$). Groups satisfying this stronger condition will be called *strongly Golod-Shafarevich.*

Golod-Shafarevich groups are known to be large in many ways:

Theorem 1.2. *Let Γ be a discrete Golod-Shafarevich group. Then its pro-p completion $\Gamma_{\widehat{p}}$ is infinite. Moreover, the following properties hold:*

a) Γ has an infinite torsion quotient.

b) If $\{D_n\Gamma\}$ is the Zassenhaus p-series of Γ, then the sequence $a_n = \log_p |\Gamma/D_n\Gamma|$ grows exponentially in n.

c) $\Gamma_{\widehat{p}}$ is not p-adic analytic.

d) $\Gamma_{\widehat{p}}$ contains a non-abelian free pro-p group.

Part a) follows from the fact that any discrete Golod-Shafarevich group has a torsion quotient which is still Golod-Shafarevich. Golod [Go] used this idea to produce first examples of infinite finitely generated residually finite torsion groups. A (standard) proof of infiniteness of Golod-Shafarevich groups (see e.g. [Ko]) automatically yields part b). Part c) follows from b) and Lazard's characterization of p-adic analytic groups (see [Lu1]), and d) is a remarkable theorem of Zelmanov [Ze1].

Among all Golod-Shafarevich groups, of particular interest are those which arise naturally in other contexts and are not just defined by a presentation satisfying the Golod-Shafarevich condition. There are two main sources of such examples: Galois groups of certain pro-p extensions of number fields (these are pro-p groups) [Ko] and fundamental groups of hyperbolic three-manifolds, discussed below in more detail. In this paper we introduce a new family of discrete Golod-Shafarevich groups which appear as lattices in certain totally disconnected locally compact groups (namely, topological Kac-Moody groups over finite fields) and, most importantly, have property (τ). The question of existence of Golod-Shafarevich groups with property (τ) was interesting for several reasons, but the main motivation came from Thurston's virtual positive Betti number conjecture in the theory of hyperbolic three-manifolds. We shall now explain the connection between the two problems.

Let Γ be the fundamental group of a compact hyperbolic three-manifold or, equivalently, a cocompact torsion-free lattice in $SO(3,1)$. Then Γ has a balanced presentation (a presentation with the same number of generators and relators) [Lu1]. Thus Γ is (strongly) Golod-Shafarevich as long as $d(\Gamma_{\widehat{p}}) \geq 5$ for some p. For any p, the condition $d(\Gamma_{\widehat{p}}) \geq 5$ can always be achieved by replacing Γ by a suitable finite index subgroup (which, of course, is also the fundamental group of some hyperbolic three-manifold) – see [LuZ] for details. In [Lu1], Lubotzky used Theorem 1.2c) to prove that Γ, if arithmetic, does not have the congruence subgroup property; the latter was a major open problem at the time, known as Serre's conjecture. Following this discovery, it seemed feasible that Golod-Shafarevich techniques could be used to attack an even more ambitious problem, Thurston's virtual positive Betti number conjecture, which asserts that Γ has a finite index subgroup with infinite abelianization. It is easy to see that Thurston's conjecture would imply Serre's conjecture. In late 80's Lubotzky and Sarnak formulated another conjecture which is weaker than Thurston's conjecture (and stronger than Serre's conjecture), with the hope that it would be more tractable, while its solution might shed some light on Thurston's conjecture.

Conjecture 1.3 (Lubotzky-Sarnak)**.** *The fundamental group of a compact hyperbolic three-manifold does not have property* (τ)*.*

Recall that a discrete group Γ has property (τ) if its finite quotients form a family of expanders, and that property (τ) is a weaker version of Kazhdan's property (T) (see Section 3 for details). Lubotzky-Sarnak conjecture follows from Thurston's conjecture since property (τ) is preserved under passage to finite index subgroups and a group with (τ) cannot map onto $\mathbb{Z}$.[2]

A purely group-theoretic approach to the Lubotzky-Sarnak conjecture was suggested by Lubotzky and Zelmanov:

Conjecture 1.4. *(Discrete) Golod-Shafarevich groups do not have property* (τ)*.*

Conjecture 1.5. *Strongly Golod-Shafarevich groups do not have property* (τ)

Conjecture 1.4 is stated in a recent paper of Zelmanov [Ze2, Conjecture B], and Conjecture 1.5 appears in [La2] where it is called the *Lubotzky-Zelmanov conjecture.* Of course, Conjecture 1.4 would imply Conjecture 1.5 which, in turn, would yield Lubotzky-Sarnak conjecture. The main goal of this paper is to provide a counterexample to Conjecture 1.4: [3]

Theorem 1.6. *For every sufficiently large prime p, there exists a finitely generated group with* (τ) *(in fact, with* (T)*), which is Golod-Shafarevich with respect to p.*

We would like to explain why Conjecture 1.4 is interesting from a purely group-theoretic point of view. A Golod-Shafarevich group is given by a presentation with a "small" set of relators and hence should have plenty of quotients. On the other hand, finite quotients of a group with property (τ) satisfy strong restrictions. Thus it was natural to expect that Golod-Shafarevich condition and property (τ) were mutually exclusive. Theorem 1.6 is therefore a rather surprising result.

We now sketch an explicit construction of Golod-Shafarevich groups with property (T). Let A be a $d \times d$ generalized Cartan matrix, let F be a finite field of characteristic p, and let $G^-_{top}(A, F)$ be the corresponding topological Kac-Moody group. The group $G^-_{top}(A, F)$ is locally compact, totally disconnected, and contains certain discrete subgroup $U^+(A, F)$ which is a lattice in $G^-_{top}(A, F)$ provided $|F| > d$, as shown by Carbone and Garland [CG] and Remy [Re3]. Dymara and Januszkiewicz [DJ] proved that if $|F| > \frac{1}{25}1764^{d-1}$, then $G^-_{top}(A, F)$ has property (T) if and only if the matrix A is 2-spherical (see Theorem 4.2). Since a lattice in a topological group G has property (T) if and only if G has property (T), it follows that $U^+(A, F)$ has property (T) whenever A is 2-spherical and $|F| > \frac{1}{25}1764^{d-1}$. An explicit presentation for $U^+(A, F)$ was found by Tits (see Theorem 4.4). While Tits' presentation uses an infinite set of generators, it can be "optimized" by removing redundant generators and corresponding relators. Under the

[2] In [LLR], it is proved that the Lubotzky-Sarnak conjecture and the geometrization conjecture imply Thurston's conjecture for all lattices in $SO(3, 1)$ which are commensurable to a lattice containing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The latter class includes all arithmetic lattices as is also shown in [LLR].

[3] To the best of my knowledge, the first person to suggest that Golod-Shafarevich groups with (τ) exist was Martin Kassabov.

above assumptions on A and F, the optimized presentation has finitely many generators and, as we show in Section 5, satisfies the Golod-Shafarevich condition (with respect to p) for certain choices of A, provided $|F| = p$.

While we explicitly construct just one family of Golod-Shafarevich groups with property (T), Theorem 1.6 automatically yields a much larger class of such groups since property (T) is preserved by quotients, and any Golod-Shafarevich group has plenty of quotients which are still Golod-Shafarevich. This idea leads to two interesting applications of Theorem 1.6 to questions seemingly unrelated to Golod-Shafarevich groups, namely examples of residually finite torsion non-amenable groups (Proposition 8.4) and examples of discrete groups with property (τ) whose profinite completion is not finitely presented (Corollary 8.6). These applications are based solely on the statement of Theorem 1.6 and not on its proof. We believe that the explicit presentations of Golod-Shafarevich groups with property (T) described in this paper as well as generalizations of our construction could yield more applications of this kind.

The paper is organized as follows. In Section 2 we introduce basic concepts arising in the definition of Golod-Shafarevich groups. Section 3 contains a very brief discussion of property (τ). Background on Kac-Moody algebras and groups is given in Section 4. In Section 5 we prove Theorem 1.6, and in Section 6 we discuss variations and generalizations of our construction. In Section 7 we give explicit examples of finitely presented Golod-Shafarevich groups with (T). Finally, Section 8 is devoted to applications of Theorem 1.6 and can be read independently of the previous three sections.

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2. Background on pro-p groups and their presentations

2.1. **Pro-p completions.** Let Γ be a group. The *pro-p topology* on Γ is given by the base of neighborhoods of identity consisting of all normal subgroups of p-power index in Γ. The completion of Γ with respect to this topology is a pro-p group called the *pro-p completion* of Γ and commonly denoted by $\Gamma_{\widehat{p}}$. The group Γ is residually-p if and only if the pro-p topology on Γ is Hausdorff or, equivalently, if the natural map $\Gamma \to \Gamma_{\widehat{p}}$ is injective.

2.2. **Zassenhaus filtration.** Let $X = \{x_1, \ldots, x_d\}$ be a finite set, and let $F = F(X)$ be the free group on X. Let $R_d := \mathbb{F}_p\langle\langle u_1, \ldots, u_d\rangle\rangle$ be the ring of non-commutative formal power series over $\mathbb{F}_p$ in d variables. The map $X \to R_d^*$ given by $x_i \mapsto 1 + u_i$ extends (uniquely) to an injective homomorphism $F \to R_d^*$, called the *Magnus embedding.* The ring R_d has natural I-adic topology where I is the ideal generated by $u_1, \dots u_d$. The closure of F in R_d^* coincides with the pro-p completion of F; this group is called the *free pro-p group* on X (below we denote it by $F_{\widehat{p}}$).

The *Zassenhaus series (filtration)* $\{D_n F_{\widehat{p}}\}_{n\geq 1}$ of the free pro-p group $F_{\widehat{p}}$ is defined as follows: for each $n \geq 1$ set $D_n F_{\widehat{p}} := \{g \in F_{\widehat{p}} \mid g \equiv 1 \mod I^n\}$. For any $g \in F_{\widehat{p}} \setminus \{1\}$ there exists a unique $n \geq 1$ such that $g \in D_n F_{\widehat{p}} \setminus D_{n+1} F_{\widehat{p}}$; we say that g has degree n and write $\deg(g) = n$. It is easy to see that

$$\deg([g,h]) \geq \deg(g) + \deg(h) \quad \text{and} \quad \deg(g^p) = p \cdot \deg(g). \tag{2.1}$$

If Γ is a finitely generated (abstract) group, the *Zassenhaus p-series* $\{D_n\Gamma\}$ can be defined as follows. If Γ is free, set $D_n\Gamma = D_n\Gamma_{\widehat{p}} \cap \Gamma$. In the general case, write Γ in the form F/N where F is a finitely generated free group, and set $D_n\Gamma := (D_nF)N/N$. It is easy to show that $D_n\Gamma$ is independent of the choice of a presentation. Alternatively, one can define $\{D_n\Gamma\}$ as the fastest descending chain of normal subgroups of Γ such that

$$D_1\Gamma = \Gamma, \quad [D_i\Gamma, D_j\Gamma] \subseteq D_{i+j}\Gamma \quad \text{and} \quad (D_i\Gamma)^p \subseteq D_{pi}\Gamma \quad \text{for} \quad i,j \geq 1.$$

2.3. **Pro-p presentations.** Once again, let X be a finite set, and let R be a subset of the free pro-p group $F(X)_{\widehat{p}}$. One says that a pro-p group G is given by the *pro-p presentation* $\langle X|R\rangle$ if $G \cong F(X)_{\widehat{p}}/N$ where N is the closed normal subgroup of $F(X)_{\widehat{p}}$ generated by R. The following properties of pro-p presentations are relevant for us (see [Lu1] for proofs):

1. Let Γ be an abstract group given by a presentation $\langle X|R\rangle$. Then the pro-p completion of Γ is given by the same presentation $\langle X|R\rangle$ considered as a pro-p presentation.

2. Let $\langle X|R\rangle$ be a (pro-p) presentation of a pro-p group G, and let $d = d(G)$ be the minimal number of generators for G.
   a) $|X| = d$ if and only if R has no relators of degree 1;
   b) G has a pro-p presentation $\langle X'|R'\rangle$ such that $|X'| = d$ and $|R'|-|R| = |X'| - |X|$.

Using these properties, we can explain why a strongly Golod-Shafarevich group is always Golod-Shafarevich. Let Γ be a discrete strongly Golod-Shafarevich group, and let $d = d(\Gamma_{\widehat{p}}) > 1$. By properties 1 and 2b), $\Gamma_{\widehat{p}}$ has a pro-p presentation $\langle X'|R'\rangle$ with $|X'| = d$ and $|R'| < d^2/4$. For each $i \geq 1$, let r_i be the number of relators of degree i in R'. By property 2a) we have $r_1 = 0$, whence $1 - dt + \sum r_i t^i < 1 - dt + \frac{d^2}{4}t^2 = (1 - \frac{dt}{2})^2$. Therefore, $1 - dt + \sum r_i t^i < 0$ for $t = 2/d$, so Γ is Golod-Shafarevich if $d > 2$. If $d = 2$, the desired inequality holds at $t = 1 - \varepsilon$ for sufficiently small $\varepsilon > 0$.

3. On properties (τ) and (T)

In this section we briefly discuss properties (τ) and (T), concentrating on the results relevant to this paper. For an excellent general introduction to the subject the reader is referred to the books [Lu2], [LuZ] and [BHV].

A locally compact group G is said to have Kazhdan's *property (T)* if the trivial representation $\mathbf{1}_G$ is an isolated point in the space of (isomorphism classes of) irreducible unitary representations of G. Property (τ), introduced by Lubotzky, is a finitary (weaker) version of property (T): one says that G has (τ) if $\mathbf{1}_G$ is isolated in the space of those irreducible unitary representations which factor through a finite quotient of G.

Now assume that G is a discrete finitely generated group. In this case property (τ) has a purely combinatorial characterization: [4]

Definition. *Let $\varepsilon > 0$. A finite graph X is called an ε-expander if for any division of the set of vertices of X into two disjoint subsets A and B, the number of edges between A and B is at least $\varepsilon \cdot \min\{|A|, |B|\}$.*

Proposition 3.1. *Let G be a discrete group generated by a finite set S. Let $\{G_i\}_{i=1}^{\infty}$ be the set of all finite quotients of G, and let S_i be the image of S in G_i. Then G has (τ) if and only if the Cayley graphs $Cay(G_i, S_i)$ form a family of ε-expanders for some $\varepsilon > 0$ (depending on S).*

Here are some elementary facts about properties (τ) and (T):

1. If a group G has (τ) (resp. (T)), then so do all its quotients.
2. The group $\mathbb{Z}$ does not have (τ).
3. If H is a finite index subgroup of G, then G has (τ) (resp. (T)) if and only if H has (τ) (resp. (T)).

These facts provide a simple and by far the most common way of showing the failure of (τ): to prove that a discrete group G does not have (τ) it suffices to find a finite index subgroup of G with infinite abelianization.

A common way to prove that a discrete group has property (T) (and hence (τ)) is to use the following criterion: a lattice in a locally compact group G has (T) if and only if G has (T). In fact, this approach was used in the seminal paper of Kazhdan [Kaz] to establish property (T) for lattices in higher-rank simple algebraic groups over local fields, e.g. $SL_n(\mathbb{Z})$ or $SL_n(\mathbb{F}_p[t])$, $n \geq 3$. [5]

4. Kac-Moody groups

4.1. **Kac-Moody algebras.** In this subsection we discuss basic properties of Kac-Moody Lie algebras (see a book of Kac [K] for more details).

A square matrix (a_{ij}) is called a *generalized Cartan matrix* if

(a) $a_{ij} \in \mathbb{Z}$ for all i, j (b) $a_{ii} = 2$ for all i

(c) $a_{ij} \leq 0$ if $i \neq j$ (d) $a_{ij} = 0 \iff a_{ji} = 0$.

For the rest of this section we fix a generalized Cartan matrix A and let d denote its size.

Let $\mathfrak{g} = \mathfrak{g}_A$ be the associated (derived)[6] Kac-Moody Lie algebra over $\mathbb{Q}$. By definition, $\mathfrak{g}$ is generated by elements $\{e_i, f_i, h_i\}_{i=1}^{d}$ satisfying the following relations:

1) $[h_i, h_j] = 0$
2) $[h_i, e_j] = a_{ij} e_j$
3) $[h_i, f_j] = -a_{ij} f_j$
4) $[e_i, f_i] = h_i$
5) $[e_i, f_j] = 0$ for $i \neq j$
6) $(\mathrm{ad}\ e_i)^{-a_{ij}+1}(e_j) = 0$ for $i \neq j$
7) $(\mathrm{ad}\ f_i)^{-a_{ij}+1}(f_j) = 0$ for $i \neq j$.

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[4] A discrete group with (T) is always finitely generated. There exist discrete groups with (τ) which are not finitely generated.

[5] The results of Kazhdan were used by Margulis in early 70's to give the first explicit construction of expanders. The concept of property (τ) was introduced much later in [Lu2].

[6] The algebra $\mathfrak{g}_A$ is the commutator subalgebra of the usual Kac-Moody algebra.

Let $\mathfrak{h}$ be the linear span of $\{h_i\}$, and let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. Then $\mathfrak{g}$ has the following root space decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x,\ h \in \mathfrak{h}\}$, and $\Delta = \{\alpha \in \mathfrak{h}^*\backslash\{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. Elements of Δ are called *roots* of $\mathfrak{g}$.

According to the defining relations, for each $1 \leq i \leq d$, the element e_i lies in some root subspace; let α_i be the corresponding root. The roots $\alpha_1, \ldots, \alpha_d$ are called simple; the set of simple roots will be denoted by Π. Each root has the form $\sum n_i\alpha_i$ where $n_i \in \mathbb{Z}$ and either $n_i \geq 0$ for all i or $n_i \leq 0$ for all i. The roots are called positive or negative accordingly; the set of positive (resp. negative) roots is denoted by Δ^+ (resp. Δ^-). If $\alpha = \sum n_i\alpha_i$, the number $\sum n_i$ is called the *height* of α and denoted by $\mathrm{ht}(\alpha)$.

4.2. **Real roots, the Weyl group and the Dynkin diagram.** For simplicity, we assume that the matrix A is *symmetrizable,* which means that A is the product of a symmetric matrix and a diagonal matrix.

Let $Q = \bigoplus_{i=1}^{d} \mathbb{Z}\alpha_i$ be the integral span of simple roots (of course, $\Delta \subset Q$). Symmetrizability of A ensures the existence of a symmetric bilinear form $(\cdot,\cdot) : Q \times Q \to \mathbb{Z}$ such that $\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} = a_{ij}$ for all i,j and $(\alpha_i,\alpha_i) > 0$ for all i. For $1 \leq i \leq d$, define the element $w_i \in \mathrm{Aut}\,(Q) \cong GL_d(\mathbb{Z})$ by setting

$$w_i(\beta) = \beta - \frac{2(\alpha_i,\beta)}{(\alpha_i,\alpha_i)}\alpha_i.$$

The group $W \subset \mathrm{Aut}\,(Q)$ generated by $\{w_i\}$ is called the Weyl group corresponding to A. The action of W clearly preserves the form $(\cdot,\cdot)$.

It is easy to see that Δ is W-invariant, that is, $W\Delta = \Delta$. Let $\Phi = W\Pi \subseteq \Delta$ be the set of W-translates of simple roots. Elements of Φ are called *real roots,* and we set $\Phi^\pm = \Delta^\pm \cap \Phi$. The remaining roots $\Delta\backslash\Phi$ are called *imaginary.* By construction, $(\alpha,\alpha) > 0$ for any $\alpha \in \Phi$. It is well known that $(\alpha,\alpha) \leq 0$ for any $\alpha \in \Delta\backslash\Phi$ (see [K, Proposition 5.2c)]).

For each $\alpha \in \Phi$ we define $w_\alpha \in \mathrm{Aut}\,(Q)$ by $w_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ (so $w_i = w_{\alpha_i}$ for $1 \leq i \leq d$). It is easy to see that if $\alpha = w\alpha_j$ for some j, then $w_\alpha = ww_jw^{-1}$.

If X is any subset of Φ, the *root subsystem* generated by X is the smallest subset Ψ of Φ such that $X \subset \Psi$ and $w_\alpha\Psi \subset \Psi$ for each $\alpha \in X$. In fact, Ψ is a root system in the "usual" sense, that is, $w_\alpha\Psi \subset \Psi$ for each $\alpha \in \Psi$.

Finally, recall the notion of the Dynkin diagram of A, which we will denote by $Dyn(A)$. We define $Dyn(A)$ to be a (multi)-graph on d vertices $\{v_1, \ldots, v_d\}$, where v_i and v_j (for $i \neq j$) are connected by $a_{ij}a_{ji}$ edges. [7]

4.3. **Kac-Moody groups.** Let F be a field. In this subsection we describe the group $G(A,F)$, the *simply-connected Kac-Moody group* over F (corresponding to A), as constructed by Tits [Ti2]. The easiest way to define $G(A,F)$ is by generators and relators. Most of the data in the presentation comes directly from the matrix A and the associated real root system $\Phi(A)$. The only external ingredient is a "Chevalley system" $\{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Phi}$ of the Kac-Moody algebra $\mathfrak{g}_A$ (see [Ti2, 3.2] and [Mo] for details). Each e_α is well defined up to sign; it is common (but not necessary) to assume that $e_{\alpha_i} = e_i$ and $e_{-\alpha_i} = f_i$ for $1 \leq i \leq d$.

[7] Our definition is different from the one in [K].

By definition, the group $G = G(A, F)$ is generated by the symbols $\{x_\alpha(u) \mid \alpha \in \Phi, u \in F\}$ satisfying relations (R1)-(R7) below. One should think of $x_\alpha(u)$ as playing the role of $\exp(ue_\alpha)$.

In all the relations u, v are elements of F (arbitrary, unless mentioned otherwise), $1 \le i, j \le d$, and $\alpha, \beta \in \Phi$.

(R1) $x_\alpha(u + v) = x_\alpha(u)x_\alpha(v)$

(R2) Let $\{\alpha, \beta\}$ be a *prenilpotent pair,* that is, there exist $w_1, w_2 \in W$ such that $w_1\alpha, w_1\beta \in \Phi^+$ and $w_2\alpha, w_2\beta \in \Phi^-$. Then

$$[x_\alpha(u), x_\beta(v)] = \prod_{i,j\ge 1} x_{i\alpha+j\beta}(C_{ij\alpha\beta}u^i v^j)$$

where the product on the right hand side is over all pairs $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $i\alpha + j\beta \in \Phi$, in some fixed order, and $C_{ij\alpha\beta}$ are integers independent of F (but depending on the order). The prenilpotency assumption implies that $|(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi| < \infty$ – see [KP, Proposition 4.7].

For $1 \le i \le d$ and $u \in F^*$ set $x_{\pm i}(u) = x_{\pm\alpha_i}(u)$, $\widetilde{w}_i(u) = x_i(u)x_{-i}(-u^{-1})x_i(u)$, $\widetilde{w}_i = \widetilde{w}_i(1)$ and $h_i(u) = \widetilde{w}_i(u)\widetilde{w}_i^{-1}$. The remaining relations are

(R3) $\widetilde{w}_i x_\alpha(u)\widetilde{w}_i^{-1} = x_{w_i\alpha}(\pm u)$

(R4) $h_i(u)x_\alpha(v)h_i(u)^{-1} = x_\alpha(vu^{2\frac{(\alpha,\alpha_i)}{(\alpha_i,\alpha_i)}})$ for $u \in F^*$

(R5) $\widetilde{w}_i h_j(u)\widetilde{w}_i^{-1} = h_j(u)h_i(u^{-a_{ji}})$

(R6) $h_i(uv) = h_i(u)h_i(v)$ for $u, v \in F^*$

(R7) $[h_i(u), h_j(v)] = 1$ for $u, v \in F^*$

The choice of a Chevalley system $\{e_\alpha\}$ determines the appropriate signs in relations (R3) and structure constants in relations (R2). In fact, the latter can be computed as follows (see [Re1, 9.2.2] for details):

Let $\{\alpha, \beta\}$ be a prenilpotent pair, and choose some order on the finite set $(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi$. Then the constants $C_{ij\alpha\beta}$ are given by the identity

$$[\exp(te_\alpha), \exp(se_\beta)] = \prod_{i,j\ge 1} \exp(C_{ij\alpha\beta}t^i s^j e_{i\alpha+j\beta}). \tag{4.1}$$

Here s and t are formal commuting variables, and (4.1) represents an equality in the ring of power series $\mathcal{U}(\mathfrak{g}_A)[[s, t]]$, where $U(\mathfrak{g}_A)$ is the universal enveloping algebra of $\mathfrak{g}_A$. In particular, $C_{11\alpha\beta}$ satisfies the equation $[e_\alpha, e_\beta] = C_{11\alpha\beta}e_{\alpha+\beta}$.

Before discussing the general structure of Kac-Moody groups, we mention two important special cases where G has an explicit realization.

1) The simplest case is when A is a matrix of finite type, that is, A is positive definite. In this case A is the Cartan matrix of a finite-dimensional simple Lie algebra $\mathfrak{g}$, and $G(A, F)$ is the group of F-points of the corresponding simply-connected Chevalley group $\mathbb{G}$. The above presentation is a slight modification of the Steinberg presentation for $\mathbb{G}(F)$.

2) Another well-known case is when A is of affine type, that is, A is positive semi-definite, but not positive-definite. For concreteness, let $d > 2$ and let A be the generalized Cartan matrix such that $a_{ij} = -1$ if $|i - j| = 1$ and $a_{ij} = 0$ if $|i - j| > 1$; in this case the Dynkin diagram is a cycle of length d, usually denoted by $\widehat{A}_{d-1}$. The Kac-Moody group $G(A, F)$ is a central extension of the group $SL_d(F[t, t^{-1}])$ by F^* .

Remark: If A is of finite type, then Δ is finite, all roots are real, and each pair $\{\alpha, \beta\}$ with $\alpha \ne -\beta$ is prenilpotent.

Next we introduce several important subgroups of G:

1. Root subgroups. For each $\alpha \in \Phi$ let $U_\alpha = \{x_\alpha(u) \mid u \in F\}$. Each U_α is isomorphic to the additive group of F.
2. "Extended" Weyl group. Let $\widetilde{W}$ be the subgroup of G generated by the elements $\{\widetilde{w}_i\}$. One can show that there exists a surjective homomorphism $\varepsilon : \widetilde{W} \to W$ such that $\varepsilon(\widetilde{w}_i) = w_i$ for $1 \le i \le d$.
3. Let $U^+ = \langle U_\alpha \mid \alpha \in \Phi^+\rangle$ and $U^- = \langle U_\alpha \mid \alpha \in \Phi^-\rangle$.
4. Let $H = \langle\{h_i(u) \mid 1 \le i \le d, u \in k\}\rangle$. It is known that relations (R6)-(R7) are defining relations for H. Thus H is isomorphic to the sum of d copies of F^*. One can think of H as the analogue of a torus.
5. Let $B^+ = \langle H, U^+\rangle$ and $B^- = \langle H, U^-\rangle$. By relations (R4), H normalizes both U^+ and U^-, so we have $B^+ = HU^+ = U^+H$ and $B^- = HU^- = U^-H$. Moreover, $U^+ \cap H = U^- \cap H = \{1\}$, as shown by Rémy [Re1, 3.5.4].
6. Finally, let $N = \langle\widetilde{W}, H\rangle$. Since $\widetilde{W}$ normalizes H, we have $N = \widetilde{W}H$. It is also easy to see that $N/H \cong W$.

Tits [Ti2] proved that (B^+, N) and (B^-, N) are BN-pairs of G. These lead to actions of G on two buildings: the positive building X^+ and the negative building X^-.

4.4. **Topological Kac-Moody groups over finite fields.** From now on we assume that F is a finite field of characteristic p. In this case the buildings X^+ and X^- are locally finite as chamber complexes. The automorphism group of a locally finite building carries a natural topology with a base of neighborhoods of identity consisting of pointwise stabilizers of finite unions of chambers.

Consider the topologies on G induced by its actions on X^+ and X^-, and let G^+_{top} and G^-_{top} be the corresponding completions of G – such groups were introduced in [RR] and called *topological* Kac-Moody groups (see also [CG] for a slightly different construction). Usually, one works with the group G^+_{top}, but for us it will be more convenient to work with G^-_{top}.

Let $\iota : G \to G^-_{top}$ be the natural map, and let Z be the kernel of ι or, equivalently, the kernel of the action of G on X^-. It is known that Z is a central subgroup of H (see [RR, 1.B, Lemma 1] which is based on [Re1, 1.5.4, 3.5.4]). In particular, $Z \cap U^- = Z \cap U^+ = \{1\}$.

Before proceeding, we shall describe the subgroups introduced above in the case $Dyn(A) = \widehat{A}_{d-1}$.

1. G^-_{top} is isomorphic to $PSL_d(F[[t^{-1}]])$. Under this isomorphism, $\iota(G)$ is mapped to $PSL_d(F[t, t^{-1}])$.
2. $\iota(B^+)$ (resp. $\iota(U^+)$) consists of matrices in $PSL_d(F[t])$ which are upper-triangular (resp. upper-unitriangular) $\mod t$.
3. $\iota(B^-)$ (resp. $\iota(U^-)$) consists of matrices in $PSL_d(F[t^{-1}])$ which are lower-triangular (resp. lower-unitriangular) $\mod t^{-1}$. The closures of these subgroups in G^-_{top} have analogous descriptions with $F[t^{-1}]$ replaced by $F[[t^{-1}]]$.
4. $\iota(H)$ consists of diagonal matrices in $PSL_d(F)$
5. $\iota(N)$ is a product of diagonal matrices in $PSL_d(F[t, t^{-1}])$ and monomial matrices.

Now we return to the general case. The group G^-_{top} has nice topological structure, strengthening the analogy with the affine case. It is easy to see that $\iota(H)$, $\iota(N)$ and $\iota(B^+)$ are discrete in G^-_{top}. Let B^-_{top} (resp. U^-_{top}) be the closures of $\iota(B^-)$ (resp. $\iota(U^-)$) in G^-_{top}. The following theorem was proved independently by Carbone and Garland [CG] and Rémy and Ronan ([RR], [Re2]):

Theorem 4.1. *The group G^-_{top} is locally compact. Moreover, B^-_{top} is an open profinite subgroup of G^-_{top}, and U^-_{top} is an open pro-p subgroup.*

As a consequence of this theorem, we see that U^- is a residually-p group, and by symmetry the same is true for U^+.

Now we turn to the discussion of when topological Kac-Moody groups have property (T). An almost complete answer to this question is given by a fundamental work of Dymara and Januszkiewicz [DJ], who established very general necessary and sufficient conditions for a group with a BN-pair to have property (T). Theorem 4.2 below is a special case of their result.

Definition. We say that the matrix A is 2-*spherical* if any pair of simple roots in $\Phi(A)$ generates a finite root subsystem.

Remark: A is 2-spherical if and only if $a_{ij}a_{ji} \leq 3$ for all $i \neq j$.

Theorem 4.2. *a) If A is not 2-spherical, G^-_{top} does not have property (T). b) If A is 2-spherical and $|F| > \frac{1}{25}1764^{d-1}$, then G^-_{top} has (T).*

The next theorem was proved independently by Carbone and Garland [CG] and Rémy [Re3]:

Theorem 4.3. *The group $\iota(B^+)$ is a non-uniform lattice in G^-_{top} provided $|F| > d$ (recall that d is the size of the matrix A).*

Now a lattice in G^-_{top} has (T) if and only if G^-_{top} has (T). Moreover, if $\iota(B^+)$ has (T), then U^+ also has (T) because (T) is invariant under the passage to a finite index subgroup. Here we use the fact that $\iota(U^+)$ has finite index in $\iota(B^+)$ and $U^+ \cap \operatorname{Ker} \iota = \{1\}$.

4.5. **Tits' presentation for U^+.** So, U^+ is always a residually-p group, and we know when U^+ has property (T). In order to test whether the pro-p completion of U^+ is a Golod-Shafarevich group, we need an explicit presentation for U^+. Such a presentation was established by Tits [Ti1, Proposition 5] (see also [Re1, 3.5.3]).

Theorem 4.4 (Tits)**.** *The group $U = U^+$ is generated by the symbols $\{x_\alpha(u) \mid \alpha \in \Phi^+, u \in F\}$ subject to relations (R1) and (R2) (see subsection 4.3).*

5. Proof of Theorem 1.6

In this section we give specific examples where U^+ is a Golod-Shafarevich group with property (T). We retain all notations from the previous section, and we will write U for U^+ (the matrix A and the field F are fixed for the entire section).

We will take F to be the prime field $\mathbb{F}_p$. This assumption is essential; in fact, we do not know if U can be a Golod-Shafarevich group when F is a non-prime field.

As before, fix an integer $d \geq 2$, and let A be the $d \times d$ generalized Cartan matrix such that $a_{ij} = -1$ for $i \neq j$. Note that $Dyn(A)$ is the complete (simply-laced) graph with d vertices. Clearly, A is 2-spherical, so by Theorem 4.2b) U has property (T) provided $p > \frac{1}{25}1764^{d-1}$.

Recall that $\Pi = \{\alpha_1, \ldots, \alpha_d\}$ denotes the set of simple roots. Finally, we set $x_\alpha := x_\alpha(1)$ for each $\alpha \in \Phi$.

5.1. **Optimizing Tits' presentation.** In this subsection we construct a new presentation for the group U which has a much smaller generating set than Tits' presentation. Informally speaking, this presentation is obtained from Tits' presentation as follows. For each non-simple positive root $\alpha \in \Phi$ and $u \in F$ we express $x_\alpha(u)$ in terms of $x_{\alpha_1}, \ldots, x_{\alpha_d}$ using Tits' presentation; then we eliminate $x_\alpha(u)$ from the generating set, which allows us to eliminate one of the defining relators in Tits' presentation as well. As a result, we obtain a presentation for U with the generating set $\{x_{\alpha_1}, \ldots, x_{\alpha_d}\}$; it will be called an optimized presentation. In the next subsection we will show that this optimized presentation satisfies the Golod-Shafarevich condition provided $d \geq 45$.

Since the matrix A is symmetric, we can normalize the form $(\cdot, \cdot)$ such that $(\gamma, \gamma) = 2$ for any $\gamma \in \Phi$. Note that $w_\alpha(\beta) = \beta - (\beta, \alpha)\alpha$ for any $\alpha, \beta \in \Phi$.

We start with a simple characterization of prenilpotent pairs.

Lemma 5.1. *A pair of positive real roots $\{\alpha, \beta\}$ is prenilpotent if and only if $(\alpha, \beta) \geq -1$.*

This result follows easily from the proof of [KP, Proposition 4.7]. For completeness we present a proof at the end of this section.

Next we determine the precise form of relations (R2) in Tits' presentation.

Lemma 5.2. *Let $\{\alpha, \beta\}$ be a prenilpotent pair of positive roots.*

a) *If $(\alpha, \beta) = -1$, then $(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi = \{\alpha + \beta\}$. Moreover, $[x_\alpha, x_\beta] = x_{\alpha+\beta}^{\pm 1}$.*

b) *If $(\alpha, \beta) \geq 0$, then $(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi = \emptyset$, whence $[x_\alpha, x_\beta] = 1$.*

Proof. a) Given $i, j \geq 1$, we have $(i\alpha + j\beta, i\alpha + j\beta) = 2(i^2 + j^2 - ij) = 2(i-j)^2 + 2ij$. If either $i > 1$ or $j > 1$, then $2(i-j)^2 + 2ij > 2$, so $i\alpha + j\beta$ cannot be a root. On the other hand, $\alpha + \beta = w_\alpha(\beta)$ is a root.

Now we prove the statement about the commutation relation. We know that $[x_\alpha, x_\beta] = x_{\alpha+\beta}^{C_{11\alpha\beta}}$ for some $C_{11\alpha\beta} \in \mathbb{Z}$, and $C_{11\alpha\beta}$ is given by the equation $[e_\alpha, e_\beta] = C_{11\alpha\beta} e_{\alpha+\beta}$ (where $\{e_\gamma\}_{\gamma\in\Phi}$ is the chosen Chevalley system). Since $(\alpha, \alpha) = (\beta, \beta) = 2$ and $(\alpha, \beta) = -1$, the root subsystem Ψ generated by α and β is finite; more precisely, $\Psi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ is of type A_2. Hence, $\{e_\gamma\}_{\gamma\in\Psi}$ is a part of a Chevalley basis for a finite-dimensional simple Lie algebra (isomorphic to $\mathfrak{sl}_3$). By [St, Theorem 1d)], $C_{11\alpha\beta} = \pm(q + 1)$ where $q \in \mathbb{Z}_{\geq 0}$ is the largest integer such that $\alpha - q\beta \in \Psi$. From the explicit description of Ψ we see that $q = 0$, so $C_{11\alpha\beta} = \pm 1$.

The proof of b) is analogous. □

If $\{\alpha, \beta\}$ is a prenilpotent pair such that $\alpha + \beta$ is a root, then after swapping α and β if necessary, we can assume that the commutator $[x_\alpha, x_\beta]$ is equal to $x_{\alpha+\beta}$, not $x_{\alpha+\beta}^{-1}$. In this case we will say that $\{\alpha, \beta\}$ is *properly ordered.*

Let P_1 be the set of properly ordered prenilpotent pairs $\{\alpha,\beta\}$ of positive roots such that $\alpha+\beta$ is a root. Let P_2 be the set of unordered prenilpotent pairs $\{\alpha,\beta\}$ of positive roots such that $\alpha\neq\beta$ and $\alpha+\beta$ is not a root. Let $P=P_1\sqcup P_2$. Now we can write Tits' presentation for U very explicitly.

Proposition 5.3. *The group U has a presentation $\langle X \mid R\rangle$ where $X=\{x_\alpha \mid \alpha\in\Phi^+\}$, and R consists of three types of relations:*

(A) $[x_\alpha,x_\beta]=x_{\alpha+\beta}$ *where* $\{\alpha,\beta\}\in P_1$.
(B) $[x_\alpha,x_\beta]=1$ *where* $\{\alpha,\beta\}\in P_2$.
(C) $x_\alpha^p=1$ *where* $\alpha\in\Phi^+$.

Proof. Recall two commutator identities: $[a,bc]=[a,c][a,b][[a,b],c]$ and $[ab,c]=[a,c][[a,c],b][b,c]$. Using these identities it is easy to see that relations (A) and (B) imply that $[x_\alpha^n,x_\beta^m]=x_{\alpha+\beta}^{nm}$ for $\{\alpha,\beta\}\in P_1$ and $[x_\alpha^n,x_\beta^m]=1$ for $\{\alpha,\beta\}\in P_2$, with $n,m\in\mathbb{Z}_{>0}$. These relations along with relations (C) form a set of defining relations for U by Theorem 4.4. $\square$

It is clear that the above generating set X is not minimal. In fact, we will see shortly that U is generated by $x_{\alpha_1},\ldots,x_{\alpha_d}$. The latter is an easy consequence of a purely combinatorial statement:

Proposition 5.4. *Let γ be a positive non-simple real root. Then there exists a pair $\{\alpha,\beta\}\in P_1$ such that $\gamma=\alpha+\beta$.*

Proof. By [CER, Lemma 6.2], there exist simple roots α_i,α_j, a (non-simple) root $\gamma_0\in\mathbb{Z}_{>0}\alpha_i+\mathbb{Z}_{>0}\alpha_j$ and $w\in W$ such that $w\alpha_i>0$, $w\alpha_j>0$ and $w\gamma_0=\gamma$. By Lemma 5.2a), $\mathbb{Z}_{>0}\alpha_i+\mathbb{Z}_{>0}\alpha_j=\{\alpha_i+\alpha_j\}$, whence $\gamma=w(\alpha_i+\alpha_j)=w\alpha_i+w\alpha_j$. Since the pair $\{\alpha_i,\alpha_j\}$ is prenilpotent, so is the pair $\{w\alpha_i,w\alpha_j\}$. Thus we can set $\{\alpha,\beta\}=\{w\alpha_i,w\alpha_j\}$ $\square$

Now we will define another presentation for U of the form $\langle X_0|R_0\rangle$ where $X_0=\{x_1,\ldots,x_d\}$ is a set of cardinality d, and x_i represents x_{α_i} in U. Informally speaking, this presentation is obtained from Tits' presentation by rewriting defining relations (A), (B) and (C) of Proposition 5.3 in terms of the new generating set X_0.

Let $F(X_0)$ be the free group on X_0. We shall define elements $C_\gamma\in F(X_0)$, for $\gamma\in\Phi^+$, by induction on $\mathrm{ht}(\gamma)$. If $\gamma=\alpha_i$ for some i, we set $C_\gamma=x_i$. Now let $\gamma\in\Phi^+$ be arbitrary, and suppose that $C_{\gamma'}$ is already defined for all γ' with $\mathrm{ht}(\gamma')<\mathrm{ht}(\gamma)$. Choose $\{\alpha,\beta\}\in P_1$ such that $\gamma=\alpha+\beta$ (this is possible by Proposition 5.4), and set $C_\gamma=[C_\alpha,C_\beta]$. The chosen pair $\{\alpha,\beta\}$ will be denoted by $p(\gamma)$. Finally, let $P_3:=\{p(\gamma)\mid\gamma\in\Phi^+\}\subset P_1$.

Define R_0 to be the subset of $F(X_0)$ consisting of the following elements:
(A) $[C_\alpha,C_\beta]C_{\alpha+\beta}^{-1}$ where $\{\alpha,\beta\}\in P_1\backslash P_3$,
(B) $[C_\alpha,C_\beta]$ where $\{\alpha,\beta\}\in P_2$,
(C) C_α^p where $\alpha\in\Phi^+$.
Note that if $\{\alpha,\beta\}\in P_3$, then $[C_\alpha,C_\beta]C_{\alpha+\beta}^{-1}=1$ (as an element of $F(X_0)$).

Let $\widetilde{U}$ be the group given by the presentation $\langle X_0|R_0\rangle$. We claim that $\widetilde{U}$ is isomorphic to U.

By construction, there is a well-defined homomorphism $\varphi:U\to\widetilde{U}$ such that $\psi(x_\alpha)=C_\alpha$ for $\alpha\in\Phi^+$ (here C_α is not an element of $F(X_0)$, but its image in $\widetilde{U}$). Now consider the homomorphism $\psi:F(X_0)\to U$ defined by $\psi(x_i)=x_{\alpha_i}$ for $1\le i\le d$. By induction on height, we see that $\psi(C_\alpha)=x_\alpha$ for $\alpha\in\Phi^+$, whence φ factors through a homomorphism $\bar{\psi}:\widetilde{U}\to U$. Clearly,

φ and $\bar{\psi}$ are mutually inverse, whence $\widetilde{U}$ is isomorphic to U. From now on the presentation $\langle X_0 \mid R_0\rangle$ of U will be referred to as the *optimized presentation.*

By construction, for each $\alpha \in \Phi^+$ the element C_α is a commutator (not necessarily left-normed) of length $\mathrm{ht}(\alpha)$ in $\{x_1, \dots, x_d\}$. Hence, the degrees of elements of R_0 (with respect to the Zassenhaus p-series of $F(X_0)$) are given as follows: $\deg([C_\alpha, C_\beta]C_{\alpha+\beta}^{-1}) = \mathrm{ht}(\alpha) + \mathrm{ht}(\beta)$ for $\{\alpha, \beta\} \in P_1\backslash P_3$, $\deg([C_\alpha, C_\beta]) = \mathrm{ht}(\alpha) + \mathrm{ht}(\beta)$ for $\{\alpha, \beta\} \in P_2$, and $\deg(C_\alpha^p) = p \cdot \mathrm{ht}(\alpha)$ for $\alpha \in \Phi^+$.

5.2. **Verifying the Golod-Shafarevich condition.** We are now ready to show that U is a Golod-Shafarevich group for sufficiently large d:

Theorem 5.5. *Suppose that $d \geq 45$ and $p > 2$. Then the optimized presentation of U satisfies the Golod-Shafarevich condition.*

Proof. Throughout this subsection t will be a fixed real number between 0 and 1. Let r_i be the number of defining relators of degree i in R_0, and let $H(t) = 1 - dt + \sum_{i=2}^{\infty} r_i t^i$ (it is clear that $r_1 = 0$). We will show that $H(2/d) < 0$ provided $d \geq 45$.

Clearly, we have

$$H(t) = 1 - dt + \sum_{\{\alpha,\beta\}\in P} t^{\mathrm{ht}(\alpha)+\mathrm{ht}(\beta)} - \sum_{\alpha\in\Phi^+\backslash\Pi} t^{\mathrm{ht}(\alpha)} + \sum_{\alpha\in\Phi^+} t^{p\cdot\mathrm{ht}(\alpha)}.$$

The third and the fourth terms in the above sum count the total weight of relators of type (A) and (B) in the optimized presentation for U; the fourth term (with the minus sign) takes into account relators from the Tits' presentation which correspond to pairs in P_3 (and become trivial after rewriting). Finally, the fifth term counts the total weight of relators of type (C).

For $i \geq 1$, let L_i be the number of real roots of height i, and for $1 \leq i \leq j$ let $L_{i,j}$ be the number of pairs $\{\alpha, \beta\} \in P$ such that $\{\mathrm{ht}(\alpha), \mathrm{ht}(\beta)\} = \{i, j\}$ as unordered pairs. Note that $L_{i,j} \leq L_i L_j$ for $i \neq j$ and $L_{i,i} \leq \binom{L_i}{2}$. Let $S(t) = \sum_{i=3}^{\infty} L_i t^i$. Then $\sum_{\{\alpha,\beta\}\in P} t^{\mathrm{ht}(\alpha)+\mathrm{ht}(\beta)} = \sum_{1\leq i\leq j} L_{i,j} t^{i+j}$, whence

$$H(t) < 1 - dt + \binom{L_1}{2} t^2 + L_{1,2}t^3 + L_{2,2}t^4 + S(t)(L_1 t + L_2 t^2) + S(t)^2/2 - L_2 t^2 - S(t) + L_1 t^p + L_2 t^{2p} + S(t^p).$$

We know that there are d simple roots, so $L_1 = d$. Since $w_j(\alpha_i) = \alpha_i + \alpha_j$ for $i \neq j$, we have $L_2 = \binom{d}{2}$. By Lemma 5.1, a pair $\{\alpha_i + \alpha_j, \alpha_k\}$ is prenilpotent if and only if $k = i$ or $k = j$. It follows that $L_{1,2} = d(d-1)$. Similarly, a pair $\{\alpha_i + \alpha_j, \alpha_k + \alpha_l\}$ is prenilpotent if and only if $k = i$, $k = j$, $l = i$ or $l = j$, so $L_{2,2} = d\binom{d-1}{2}$. Therefore, we have

$$H(t) < 1 - dt + d(d-1)t^3 + d\binom{d-1}{2}t^4 + S(t)(dt + \binom{d}{2}t^2) + S(t)^2/2 - S(t) + dt^p + \binom{d}{2}t^{2p} + S(t^p).$$

Now set $S = S(2/d)$. Using the trivial estimate $S(t^p) < S(t)^p$ and assuming that $p > 2$, we get

$$H(2/d) < 1 - 2 + \frac{8}{d} + \frac{8}{d} + S(2+2) + S^2/2 - S + S^p = -1 + \frac{16}{d} + 3S + S^2/2 + S^p.$$

To prove Theorem 5.5, it suffices to show that $S < 1/5$ for $d \geq 45$.

In order to estimate S, we introduce the notion of the depth of a root:

Definition. Let β be a positive real root.
a) Set $N(\beta) = \{\gamma \in \Phi^+ \mid \gamma = w_i\beta \text{ for some } i \text{ and } \mathrm{ht}(\gamma) > \mathrm{ht}(\beta)\}$.
b) The *depth* of β is the minimal length l of a sequence of real roots $\beta_1, \ldots, \beta_l$ where β_1 is simple, $\beta_l = \beta$ and $\beta_{j+1} \in N(\beta_j)$ for $1 \leq j \leq l-1$.

To prove that depth is always defined, we need to show that for every $\beta \in \Phi^+$ there exists i such that $\mathrm{ht}(w_i\beta) < \mathrm{ht}(\beta)$. Suppose this is not the case for some β. Then $(\beta, \alpha_i) \leq 0$ for all i, and since β is a linear combination of α_i's with non-negative coefficients, it would follow that $(\beta, \beta) \leq 0$. The latter is impossible since β is real.

For each $i \geq 1$, let Φ_i be the set of real roots of depth i and let $S_i(t) = \sum_{\alpha \in \Phi_i} t^{\mathrm{ht}(\alpha)}$. Clearly, roots of depth 1 (resp. 2) are precisely roots of height 1 (resp. 2), so $S(t) = \sum\limits_{i \geq 3} S_i(t)$.

Lemma 5.6. *The following hold:*
a) Roots of depth 3 are all of the form $\alpha_i + \alpha_j + 2\alpha_k$ where i, j, k are distinct, so $S_3(t) < \frac{d^3}{2} t^4$.
b) For every $i \geq 3$, $S_{i+1}(t) < S_i(t)(2t + dt^3)$.

The required estimate on S follows immediately from the above lemma. Indeed, we have

$$S(2/d) = \sum_{i=3}^{\infty} S_i(2/d) < \frac{S_3(2/d)}{1 - 4/d - 8/d^2} < \frac{8}{d - 4 - 8/d},$$

so $S(2/d) < 1/5$ for $d \geq 45$. This completes the proof of Theorem 5.5 □

Proof of Lemma 5.6. a) This is straightforward.
b) Clearly, $\Phi_{i+1} = \bigcup\limits_{\alpha \in \Phi_i} N(\alpha)$ for every $i \geq 1$. Therefore, it is enough to show that for every α of depth at least 3, $\sum\limits_{\beta \in N(\alpha)} t^{\mathrm{ht}(\beta)} < t^{\mathrm{ht}(\alpha)}(2t + dt^3)$. Since $N(\alpha)$ has at most d elements, the last inequality will follow if we show that $N(\alpha)$ contains at most two roots β with $\mathrm{ht}(\beta) \leq \mathrm{ht}(\alpha) + 2$.

Write α in the from $\sum_{i=1}^d n_i\alpha_i$. For $1 \leq j \leq d$, we have $w_j(\alpha) = \alpha + (\sum_{i \neq j} n_i - 2n_j)\alpha_j$. Since $\mathrm{ht}(\alpha) = \sum_{i=1}^d n_i$, we have $\mathrm{ht}(w_j\alpha) = 2\mathrm{ht}(\alpha) - 3n_j$. Hence $\mathrm{ht}(w_j\alpha) \leq \mathrm{ht}(\alpha) + 2$ if and only if $3n_j \geq \mathrm{ht}(\alpha) - 2$. The latter inequality holds for at most 3 values of j (provided $\mathrm{ht}(\alpha) > 2$): if $\mathrm{ht}(\alpha) > 8$, this follows from an obvious counting argument, and if $3 \leq \mathrm{ht}(\alpha) \leq 8$, this is an easy case-by-case verification.

Thus we showed that there are at most 3 roots of the form $w_j\alpha$ whose height is at most $\mathrm{ht}(\alpha) + 2$. However, one of those 3 roots has height smaller than $\mathrm{ht}(\alpha)$ and hence does not lie in $N(\alpha)$. This observation finishes the proof. □

Remark: In the course of the proof of Theorem 5.5 we showed that the optimized presentation for U^+ has no relators of degree 2.

Proof of Lemma 5.1. First, assume that $(\alpha, \beta) \geq -1$. Since $\alpha, \beta \in \Phi^+$, to prove that $\{\alpha, \beta\}$ is prenilpotent it suffices to show that there is $w \in W$ such that $w\alpha, w\beta \in \Phi^-$. If $(\alpha, \beta) = 0$, then $w_\alpha w_\beta(\alpha) = w_\alpha(\alpha) = -\alpha$,

$w_\alpha w_\beta(\beta) = -\beta$, so we take $w = w_\alpha w_\beta$. If $(\alpha, \beta) \geq 1$, then $w_\beta(\alpha) + w_\alpha(\beta) = (1 - (\alpha, \beta))(\alpha + \beta) \in \bigoplus_{i=1}^{d} \mathbb{Z}_{\leq 0}\alpha_i$. Thus $w_\beta(\alpha) \in \Phi^-$ or $w_\alpha(\beta) \in \Phi^-$, so both α and β are mapped to Φ^- by w_β or w_α. Finally, if $(\alpha, \beta) = -1$, we have $\{w\alpha, w\beta\} = \{-\beta, -\alpha\}$ for $w = w_\alpha w_\beta w_\alpha$.

Now suppose that $(\alpha, \beta) \leq -2$, and set $q = -(\alpha, \beta)$. It is well known that the set $\Omega := \{k \in \mathbb{Z} \mid \alpha + k\beta \in \Delta\}$ consists of consecutive integers (without gaps) – see [K, Proposition 3.6 b)]. Since $w_\alpha(\beta) = \beta + q\alpha \in \Phi$, we have $q \in \Omega$. We also know that $0 \in \Omega$, whence $1 \in \Omega$, that is, $\alpha + \beta \in \Delta$. Since $(\alpha + \beta, \alpha + \beta) = 4 - 2q \leq 0$, $\alpha + \beta$ is an imaginary root. This implies that $\{\alpha, \beta\}$ is not prenilpotent. Indeed, if $w \in W$ is such that $w\alpha, w\beta \in \Phi^-$, then $w(\alpha + \beta) \in \Delta^-$. The latter is impossible since the set of positive imaginary roots is W-invariant by [K, Proposition 5.2a)]. $\square$

6. Variations and generalizations

In this section we discuss possible variations and generalizations of our construction, which could yield a larger class of (explicitly defined) Golod-Shafarevich groups with property (T) as well as strengthening of the statement of Theorem 1.6. Potential applications of such generalizations will be discussed in Section 8.

6.1. **Using other generalized Cartan matrices.** The assumption that $Dyn(A)$ is a complete graph was made to keep computations as simple as possible, and we could work in much more general situation. For instance, everything in subsection 5.1 remains valid as long as $Dyn(A)$ is simply-laced, i.e. $a_{ij} = 0$ or -1 for all $i \neq j$. The essential thing for U^+ to be Golod-Shafarevich is that A should have few zero entries (as each such entry leads to a defining relator of degree 2).

In general, when A is 2-spherical, there is a presentation for U^+ which has fewer relations than Tits' presentation – see [AM, Théorème A] and [Cap, Theorem 3.6]. Using this presentation and a more careful counting argument, one can show that in the case when $Dyn(A)$ is a complete graph on d vertices, U^+ is a Golod-Shafarevich group for $d \geq 12$.

6.2. **Finitely presented Golod-Shafarevich groups with (T).** Abramenko [A] proved that, assuming $|F| > 6$, $U^+ = U^+(A, F)$ is finitely presented if and only if A is 3-spherical, that is, every triple of simple roots in $\Phi(A)$ generates a finite root subsystem.

Now assume that $Dyn(A)$ is simply-laced. Then it is easy to see that A is 3-spherical if and only if $Dyn(A)$ does not contain cycles of length three. Thus if $Dyn(A)$ is a complete graph, then U^+ is not finitely presented.

Nevertheless, it is easy to construct a finitely presented Golod-Shafarevich group with property (T). One way is to use the following result of Shalom (see [Sh, Theorem 6.7]):

Theorem 6.1. *Let Γ be a discrete group with property (T). Let $\langle X|R\rangle$ be a presentation for Γ with $|X| < \infty$. Then there exists a finite subset R' of R such that the group $\Gamma' := \langle X|R'\rangle$ also has property (T).*

Clearly, if Γ is a Golod-Shafarevich group, then so is Γ'. The disadvantage of this approach is that we do not have an explicit presentation for Γ' and we do not know if Γ' is residually finite.

Another way to construct a finitely presented Golod-Shafarevich group with property (T) is as follows. Let A be a $d\times d$ generalized Cartan matrix such that $Dyn(A)$ is the complete bipartite graph $K_{d/2,d/2}$ (d - even). Then A is 3-spherical, so U^+ is a finitely presented group with property (T). While U^+ is not a Golod-Shafarevich group, it does come very close to satisfying the required inequality (provided F is a prime field). It turns out that U^+ has a subgroup of index p which is Golod-Shafarevich when d is large enough.

Moreover, if A is 3-spherical, an explicit finite presentation for U^+ is given by the following theorem of Devillers and Mühlherr [DM, Corollary 1.2]: [8]

Theorem 6.2. *Suppose that A is 3-spherical, $|F|\geq 16$, and $U=U^+(A,F)$. Let $\{\alpha_1,\ldots,\alpha_d\}$ be the simple roots in $\Phi(A)$. Let $U_i=U_{\alpha_i}$ for $1\leq i\leq d$ and $U_{i,j}=\langle U_{\alpha_i},U_{\alpha_j}\rangle$ for $1\leq i<j\leq d$. Then U is the amalgamated product of the system $\{U_i\}\cup\{U_{i,j}\}$ with respect to inclusions $U_i\to U_{i,j}$ and $U_j\to U_{i,j}$.*

Remark: The conclusion of Theorem 6.2 can be reformulated as follows. For $1\leq i\leq d$ choose some generating set X_i of U_i, and for $1\leq i<j\leq d$ choose some presentation $\langle X_i\cup X_j\,|\,R_{i,j}\rangle$ of $U_{i,j}$. Then U is given by the presentation $\langle X|R\rangle$ where $X=\bigcup\limits_{1\leq i\leq d} X_i$ and $R=\bigcup\limits_{1\leq i<j\leq d} R_{i,j}$.

Thus the second method produces a residually-p Golod-Shafarevich group with (T) given by an explicit finite presentation. The details of this construction are given in the next section.

6.3. Dependence of the number of generators on p.

Question 6.3. *For which pairs (p,d) (p-prime, $d\geq 2$) does there exist a group Γ with property (T) which is Golod-Shafarevich with respect to p and such that $d(\Gamma_{\widehat{p}})=d$?*

In the previous section we showed that such a group exists for $d\geq 45$ and $p>\frac{1}{25}1764^{d-1}$. An interesting problem is to remove the dependence of d on p, e.g. to produce such groups for pairs (p,d) where p is fixed and $d\to\infty$ (at least for large enough p). A natural thing to try would be to start with one of our examples and then take a suitable finite index subgroup. In this way, one can always ensure that the number of generators is as large as needed, but it is hard to control if the obtained group will be Golod-Shafarevich.

Of course, it is possible that the group $U^+(A,F)$ always has (T) whenever A is 2-spherical and $Dyn(A)$ is simply-laced [9] - if this is true, the above problem would be solved. However, proving that $U^+(A,F)$ has (T) when $|F|$ is small with respect to the size of A, would require new ideas.

Finally, we should mention that a solution to the above problem would be a major step towards proving the conjecture that a Golod-Shafarevich group always has an infinite quotient with property (T) (see Conjecture 8.2).

7. Finitely presented examples. Explicit construction.

In this section we will explicitly construct a finitely presented Golod-Shafarevich group with property (T). We believe that our technique could be useful in the future for construction of Golod-Shafarevich groups with various additional properties (e.g. see Question 6.3).

[8] Abramenko informed the author that the proof of his finite presentability criterion yields the same result in 3-spherical case, but the techniques are completely different.

[9] Abramenko showed that if $|F|=2$ or 3, then $U^+(A,F)$ may not be finitely generated even if A is 2-spherical. However, this cannot happen if $Dyn(A)$ is simply-laced.

7.1. **Some basic facts about presentations.**

1. Presentations of finite index subgroups. Suppose that a group G is given by some presentation $\langle X|R\rangle$. Let F be the free group on X, and let N be the normal closure of R in F (so that $G \cong F/N$). Let H be a finite index subgroup of G. Let F_1 be the preimage of H under the natural map $F \to G$, let Y be a free generating set for F_1, and let T be a transversal for F_1 in F. Then H is given by the presentation

$$\langle Y \mid \{r^t\}_{r\in R,\, t\in T}\rangle \text{ where } r^t = t^{-1}rt. \tag{7.1}$$

2. Generator eliminations. Let $\langle X|R\rangle$ be a presentation for a group G where $X = \{x_1, \ldots, x_k\}$, and suppose that one of the defining relations has the form $x_i = w_i$ where the word w_i does not depend on x_i. Then one can construct a new presentation for G be eliminating the generator x_i, the relation $x_i = w_i$, and substituting w_i for x_i in all other relations. We shall call such procedure a *generator elimination.*

3. Elementary transformations. Let $\langle X|R\rangle$ be some presentation where $R = \{r_1, \ldots, r_s\}$. Fix $i, j \in \{1, \ldots, s\}$ with $i \neq j$, and let $R' = \{r'_1, \ldots, r'_s\}$ where $r'_k = r_k$ for $k \neq i$ and $r'_i = r_i r_j^{\pm 1}$ or $r'_i = r_j^{\pm 1} r_i$. Then the presentations $\langle X|R\rangle$ and $\langle X|R'\rangle$ define isomorphic groups. We will say that R' is obtained from R by an *elementary transformation.*

7.2. **The construction.** Fix $d \in \mathbb{N}$, and let $A = (a_{ij})_{i,j=1}^{2d}$ be the $2d \times 2d$ generalized Cartan matrix such that $a_{ij} = 0$ if $i - j$ is even (and $i \neq j$), and $a_{ij} = -1$ if $i - j$ is odd. Then $Dyn(A)$ is isomorphic to the complete simply-laced bipartite graph $K_{d,d}$. In particular, $Dyn(A)$ does not contain any cycles of length 3, so A is 3-spherical.

Let $\{\alpha_1, \ldots, \alpha_{2d}\}$ be the simple roots in $\Phi(A)$. Given $i, j \in \{1, \ldots, 2d\}$ let $\Phi_{i,j}$ be the root subsystem of $\Phi(A)$ generated by α_i, α_j. Then $\Phi_{i,j}$ is a root system of type $A_1 \times A_1$ if $i - j$ is even, and A_2 if $i - j$ is odd.

Let $p > 16$ and $U = U^+(A, \mathbb{F}_p)$. Let $U_i = U_{\alpha_i}$ for $1 \leq i \leq 2d$ and $U_{i,j} = \langle U_{\alpha_i}, U_{\alpha_j}\rangle$ for $1 \leq i < j \leq 2d$. Clearly, $U_{i,j} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ if $i - j$ is even, and $U_{i,j}$ is isomorphic to the upper-unitriangular subgroup of $SL_3(\mathbb{F}_p)$ if $i-j$ is odd. Hence, $U_{i,j} = \langle x_{\alpha_i}, x_{\alpha_j} \mid x_{\alpha_i}^p = x_{\alpha_j}^p = [x_{\alpha_i}, x_{\alpha_j}] = 1\rangle$ if $i - j$ is even and $U_{i,j} = \langle x_{\alpha_i}, x_{\alpha_j} \mid x_{\alpha_i}^p = x_{\alpha_j}^p = [[x_{\alpha_i}, x_{\alpha_j}], x_{\alpha_j}] = [[x_{\alpha_j}, x_{\alpha_i}], x_{\alpha_i}] = 1\rangle$ if $i - j$ is odd.

For convenience, we introduce shortcut notations for the generators of U: for $1 \leq i \leq d$ we set $x_i = x_{\alpha_{2i-1}}$ and $y_i = x_{\alpha_{2i}}$. By Theorem 6.2, U is given by the presentation $\langle X|R\rangle$ where $X = \{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ and R is the following set of relations

$(A_{i,j})$	$[x_i, x_j] = 1,\ 1 \leq i < j \leq d$	$(B_{i,j})$	$[y_i, y_j] = 1,\ 1 \leq i < j \leq d$
$(C_{i,j})$	$[[x_j, y_i], y_i] = 1,\ 1 \leq i, j \leq d$	$(D_{i,j})$	$[[y_i, x_j], x_j] = 1,\ 1 \leq i, j \leq d$
(E_i)	$x_i^p = 1,\ 1 \leq i \leq d$	(F_i)	$y_i^p = 1,\ 1 \leq i \leq d$

Let F be the free group on X, and let F' be the unique normal subgroup of F of index p which contains all the generators in X except x_1. Let $\pi : F \to U$ be the natural quotient map and $V = \pi(F')$. We will show that V is a Golod-Shafarevich group when $d \geq 300$ by starting with a presentation given by (7.1), then performing several generator eliminations followed by suitable elementary transformations and elimination of redundant relations.

It is clear that F' is freely generated by the set

$$X' = \{z\} \cup \{x_{i,j} : 2 \le i \le d, 0 \le j \le p-1\} \cup \{y_{i,j} : 1 \le i \le d, 0 \le j \le p-1\}$$

where $z = x_1^p, x_{i,j} = x_i^{x_1^j}$ and $y_{i,j} = y_i^{x_1^j}$. The set $T = \{1, x_1, \dots x_1^{p-1}\}$ is a transversal for F_1 in F. Therefore, V is given by the presentation $\langle X'|R'\rangle$ where $R' = R^T$ is the set of T-conjugates of relations (A)-(F) above.

7.3. **Elimination of generators.**

1. The relation (E_1) can be written as $z = 1$. Note that (E_1) is T-invariant, that is, $z^t = z$ for any $t \in T$, whence T-conjugates of (E_1) do not yield new relations. Thus we can eliminate the generator z and all relations in $(E_1)^T$, and replace z by 1 in all other relations.

2. For each $i = 2, \dots, d$, the relation $(A_{1,i})$ can be written as $x_i^{x_1} = x_i$ or, equivalently, $x_{i,1} = x_{i,0}$. The T-conjugates of this relation are

$$x_{i,1} = x_{i,0},\ x_{i,2} = x_{i,1}, \dots, x_{i,p-1} = x_{i,p-2},\ x_{i,0} = x_{i,p-1}.$$

Thus we can eliminate all relations in $(A_{1,i})^T$ as well as the generators $x_{i,1}, \dots x_{i,p-1}$ (replacing them by $x_{i,0}$ in other relations).

3. The final source of generator eliminations is relations $\{(D_{i,1})\}$. For each $i = 1, \dots, d$, we can write the relation $(D_{i,1})$ as $(y_i^{-1}(y_i)^{x_1})^{x_1} = y_i^{-1}(y_i)^{x_1}$ or, equivalently, $y_{i,2} = y_{i,1}y_{i,0}^{-1}y_{i,1}$. The T-conjugates of $(D_{i,1})$ are

$$y_{i,n+2} = y_{i,n+1}y_{i,n}^{-1}y_{i,n+1}, \quad 0 \le n \le p-1$$

(where subscripts are taken modulo p). The first $p-2$ of these T-conjugates can be used to eliminate $y_{i,2}, y_{i,3}, \dots, y_{p-1}$. We get

$$y_{i,n} = (y_{i,1}y_{i,0}^{-1})^{n-1}y_{i,1} \quad \text{for } 2 \le n \le p-1 \tag{7.2}$$

The last two T-conjugates of $(D_{i,1})$ are easily seen to be equivalent to the relation $(y_{i,1}y_{i,0}^{-1})^p = 1$. However, this relation is a consequence of other relations since $y_{i,0}^p = y_{i,1}^p = 1$ by (F_i) and its T-conjugates, and $[y_{i,1}, y_{i,0}] = 1$ by $(C_{i,1})$ (see below).

7.4. **Elementary transformations.** From now on we write $x_i = x_{i,0}$, $y_i = y_{i,0}$ and $z_i = y_{i,1}$. Elimination of generators in 7.3 yields a new presentation $\langle X''|R''\rangle$ of V where $X'' = \{x_i\}_{i=2}^d \cup \{y_i\}_{i=1}^d \cup \{z_i\}_{i=1}^d$ and R'' is the union of the following subsets of R':

$$R(A) = \bigcup_{2\le i<j\le d} (A_{i,j})^T, \qquad R(D) = \bigcup_{1\le i\le d,\ 2\le j\le d} (D_{i,j})^T,$$

$$R(B) = \bigcup_{1\le i<j\le d} (B_{i,j})^T, \qquad R(E) = \bigcup_{2\le i\le d} (E_i)^T,$$

$$R(C) = \bigcup_{1\le i\le d,\ 2\le j\le d} (C_{i,j})^T, \qquad R(F) = \bigcup_{1\le i\le d} (F_i)^T.$$

$$R_1(C) = \bigcup_{1\le i\le d} (C_{i,1})^T,$$

(all these relations are rewritten in terms of the new generating set X'').

We shall apply suitable elementary transformations to each of the sets $(A_{i,j})^T$, $(B_{i,j})^T, \dots, (F_i)^T$ (as described below) and then eliminate redundant relations. The new sets of relations will be denoted $(A_{i,j})^T_*, (B_{i,j})^T_* \dots$ $(F_i)^T_*$, and let R_* denote the union of those sets.

If S is any subset of R_*, we set $H_S(t) = \sum_{r\in S} t^{\deg(r)}$. By the degree of a relation $r_1 = r_2$ (where r_1, r_2 are elements of the free group $F(X'')$) we mean the degree of $r_1 r_2^{-1}$ with respect to the Zassenhaus p-series on $F(X'')$.

1. We start with the set $R(C)$. Let $1 \le i, j \le d$ with $j > 1$. The relation $[[x_j, y_i], y_i] = 1$ has degree 3, and so do all of its T-conjugates. Notice, however, that all of these conjugates will depend on just 3 variables: x_j, y_i and z_i. After applying elementary transformations to the set $(C_{i,j})^T$, we can assume that the number of relations of degree n in the transformed set $(C_{i,j})^T_*$ does not exceed the number of left-normed commutators of length n in 3 variables, which is surely less than 3^n. Thus

$$H_{(C_{i,j})^T_*}(t) \le \sum_{n=3}^{\infty} 3^n t^n = \frac{27t^3}{1-3t} \text{ for } 1 \le i \le d \text{ and } 2 \le j \le d$$

2. The set $R_1(C)$. For $1 \le i \le d$, the relation $(C_{i,1})$ is equivalent to $[y_i, z_i] = 1$. By (7.2), all T-conjugates of $(C_{i,1})$ are products of commutators in y_i and z_i, so they are consequences of $(C_{i,1})$. Therefore, $H_{(C_{i,1})^T_*}(t) = t^2$ for $1 \le i \le d$.

3. The set $R(D)$ is treated in the same way as $R(C)$. We conclude that $H_{(D_{i,j})^T_*}(t) \le \frac{27t^3}{1-3t}$ for $1 \le i \le d$ and $2 \le j \le d$.

4. The set $R(A)$. For each i, j, with $2 \le i < j \le d$, the relation $[x_i, x_j] = 1$ is T-invariant, so its T-conjugates do not yield new relations. Therefore $H_{(A_{i,j})^T_*}(t) = t^2$ for $2 \le i < j \le d$.

5. The set $R(E)$. As in the previous case, for each $i > 1$, the relation $x_i^p = 1$ is T-invariant. Thus, $H_{(E_i)^T_*} = t^p$ for $2 \le i \le d$.

6. The set $R(F)$. For $1 \le i \le d$, T-conjugates of the relation $y_i^p = 1$ are $y_{i,n}^p = 1$, $0 \le n \le p-1$. Since $y_{i,n} = (z_i y_i^{-1})^{n-1} z_i$ and $[y_i, z_i] = 1$, the relations $y_{i,n}^p = 1$ for $n \ge 2$ are redundant. Hence, $H_{(F_i)^T_*} = 2t^p$ for $1 \le i \le d$.

7. The set $R(B)$. This case requires a more delicate argument. Let $1 \le i < j \le d$. By (7.2), the T-conjugates of the relation $[y_i, y_j] = 1$ are

$$[(z_i y_i^{-1})^{n-1} z_i, (z_j y_j^{-1})^{n-1} z_j] = 1, \quad 0 \le n \le p-1.$$

Clearly, all these conjugates depend only on four variables y_i, y_j, z_i, z_j, so after applying elementary transformations we can assume that there are at most 4^n relations of degree n for $n \ge 2$. However, we need a better bound for $n = 2$.

By basic commutator identities, for each $n = 0, \dots, p-1$, we have

$$[(z_i y_i^{-1})^{n-1} z_i, (z_j y_j^{-1})^{n-1} z_j] = [z_i, z_j]^{n^2} ([z_i, y_j][y_i, z_j])^{-n(n-1)} [y_i, y_j]^{(n-1)^2} P_3(n)$$

where $P_3(n)$ is a product of commutators of length at least 3. Hence, if $F_{i,j}$ is the free group on $\{y_i, y_j, z_i, z_j\}$, then the projection of the set $(B_{i,j})^T$ to $\gamma_2 F_{i,j} / \gamma_3 F_{i,j}$ lies in the linear span of just three elements: $[y_i, y_j]$, $[z_i, z_j]$ and $[y_i, z_j][z_i, y_j]$. Therefore, we can assume that there are at most 3 relations of degree 2 in the transformed set $(B_{i,j})_*^T$. Arguing as in previous cases, we have $H_{(B_{i,j})_*^T} \leq 3t^2 + \frac{64t^3}{1-4t}$ for $1 \leq i < j \leq d$.

Finally, let $H(t) = 1 - |X''|t + H_{R_*}(t)$. Combining the above estimates, we get

$$H(t) \leq 1-(3d-1)t+d(d-1)\frac{27t^3}{1-3t}+dt^2+d(d-1)\frac{27t^3}{1-3t}+\binom{d-1}{2}t^2+$$

$$(d-1)t^p+d\cdot 2t^p+\binom{d}{2}\left(3t^2+\frac{64t^3}{1-4t}\right) \leq 1-(3d-1)t+2d^2t^2+3dt^p+\frac{86d^2t^3}{1-4t}.$$

It is straightforward to check that $H(\frac{2}{3d}) < 0$ for $d \geq 300$, so the presentation $\langle X''|R_*\rangle$ satisfies the Golod-Shafarevich condition.

8. Applications

Since any Golod-Shafarevich group has "plenty" of infinite quotients, Theorem 1.6 provides a large supply of residually finite groups with property (τ); moreover, these groups are very different from classical examples of groups with (τ) such as $SL_n(\mathbb{Z})$ or $SL_n(\mathbb{F}_p[t])$. In this section we discuss some consequences of Theorem 1.6 and its potential applications in the future.

8.1. Theorem 1.6 provides further indication that groups with property (τ) are not necessarily "small from the top". Some evidence in this direction was recently obtained in [Ka] and [JKN].

First, results of Kassabov [Ka] already imply the existence of a "large" pro-p group with property "(τ)". Of course, a compact group always has property (τ) in the usual sense. However, if G is a finitely generated profinite group and S is a finite generating set for G, one can say that the pair (G, S) has "(τ)" if finite quotients of G form a family of expanders (with respect to the generating sets which are images of S). Kassabov showed that property "(τ)" in this sense holds for the pair $(EL_n(\mathbb{F}_p\langle\langle x_1, \ldots x_d\rangle\rangle), S)$, $n \geq 3$, where $\mathbb{F}_p\langle\langle x_1, \ldots x_d\rangle\rangle$ is the ring of formal power series over $\mathbb{F}_p$ in non-commuting variables $\{x_1, \ldots, x_d\}$, and S is the set of elementary matrices $\{1 + x_k e_{ij}\}$. Since the group $EL_n(\mathbb{F}_p\langle\langle x_1, \ldots x_d\rangle\rangle)$ has a finite index pro-p subgroup, one immediately obtains the following interesting consequence.

Corollary 8.1. [10] *There exists a family of expanders $\{G_n\}$ (with the bounded number of generators) such that each G_n is a finite p-group, the sequence $\log_p|G_n|$ is exponential in n while the exponent and nilpotency class of G_n are linear in n.*

Our results provide new examples of such families. Indeed, if Γ is any Golod-Shafarevich group with property (τ), its finite p-quotients satisfy the conclusion of Corollary 8.1 by Theorem 1.2b).

In [JKN], it is shown that a group with property (τ) can have very large (almost exponential) subgroup growth; moreover, it is possible that the method can be extended to produce groups with property (τ) of exponential or even super-exponential subgroup growth. We should point out that the groups constructed in that paper are very far from being residually-p, so it is still possible that if Γ is a discrete group with (τ), then the subgroup growth of the pro-p completion of Γ cannot be too large. In this context, it would

[10] We do not know if such example was known before [Ka].

be interesting to determine subgroup growth for the pro-p completions of our examples.

8.2. Next we discuss how Theorem 1.6 can be used to construct large families of groups with property (T).

The following conjecture was suggested by Lubotzky.

Conjecture 8.2. *Every Golod-Shafarevich group has an infinite quotient with property* (T).

Conjecture 8.2 was originally motivated by the following well-known question asked in slightly different forms by Vershik [Ve] and de la Harpe [H] (see also [Ze2]):

Question 8.3. *Does there exist an amenable Golod-Shafarevich group?*

Indeed, an infinite group with property (T) cannot be amenable, so Conjecture 8.2 would imply the negative answer to Question 8.3.

Conjecture 8.2 is also interesting in its own right and, if true, would provide further testimony to the "largeness" of Golod-Shafarevich groups. Lubotzky proposed the following strategy for using the results of this paper to attack Conjecture 8.2. Let $\Gamma = \langle X \mid R\rangle$ be a Golod-Shafarevich group. Suppose, one can construct a group $\Gamma_1 = \langle X \mid R_1\rangle$ (with the same set of generators as Γ) such that Γ_1 has (T) and the group $\Gamma_2 = \langle X \mid R \cup R_1\rangle$ is Golod-Shafarevich. Then Γ_2 is an infinite quotient of Γ, and Γ_2 has (T) being a quotient of Γ_1.

At the moment, the supply of Golod-Shafarevich groups with (T) is insufficient to make the above strategy work. One should either construct a more general class of Golod-Shafarevich groups with (T) or find a way to prove that the group Γ_2 defined above is infinite without showing that it is Golod-Shafarevich.

While Conjecture 8.2 is out of reach for the moment, we can still give an interesting application of Theorem 1.6 which uses the above ideas.

Proposition 8.4. *There exists an infinite residually finite torsion group with property (T).*

Proof. Let Γ be a Golod-Shafarevich group with property (T). As mentioned in the introduction, Γ has a p-torsion quotient Γ_0 which is still Golod-Shafarevich. Indeed, let $g_1, g_2, \ldots$ be elements of Γ listed in some order. Given a sequence of positive integers $n_1, n_2, \ldots$, let Γ_0 be the quotient of Γ by the normal subgroup generated by $\{{g_k}^{p^{n_k}}\}_{k=1}^{\infty}$. Clearly, Γ_0 is Golod-Shafarevich provided the numbers $\{n_k\}$ are large enough.

Being a quotient of Γ, the group Γ_0 has property (T). Even though Γ_0 may not be residually finite, the image of Γ_0 in its pro-p completion is infinite. The latter group clearly satisfies the required conditions. □

An immediate corollary of Proposition 8.4 answers a question of de la Harpe [H2, Question 5]. We are grateful to Rostislav Grigorchuk for suggesting this application.

Corollary 8.5. *There exists a residually finite torsion non-amenable group.*

8.3. The following Corollary of Theorem 1.6 gives an affirmative answer to a question of Lubotzky and Zuk [LuZ, Question 1.27].

Corollary 8.6. *There exists a finitely generated group with property* (τ) *whose profinite completion is not finitely presented.*

Proof. Let Γ_0 be a p-torsion Golod-Shafarevich group with property (T) (constructed as in the proof of Proposition 8.4). Since Γ_0 is Golod-Shafarevich, it is easy to see that there exist a sequence of positive integers $n_1 < n_2 < \ldots$ and a sequence of elements $g_1, g_2, \ldots$ of Γ_0 such that for each i we have

a) $g_i \in \gamma_{n_i}\Gamma_0$,

b) $g_i \notin \langle g_1, \ldots, g_{i-1}\rangle^{\Gamma_0}\gamma_{n_i+1}\Gamma_0$,

c) the group $\Gamma_1 := \Gamma_0/\langle\{g_i\}_{i=1}^{\infty}\rangle^{\Gamma_0}$ is Golod-Shafarevich.

As usual, $\gamma_k\Gamma_0$ denotes the k^{th} term of the lower central series of Γ_0, and $\langle S\rangle^{\Gamma_0}$ is the normal subgroup of Γ_0 generated by S.

We claim that the group Γ_1 satisfies the conclusion of Corollary 8.6. Indeed, Γ_1 has property (T) as a quotient of Γ_0. If G_1 denotes the profinite completion of Γ_1, then G_1 is also the pro-p completion of Γ_1 since every finite quotient of Γ_1 is a p-group. Finally, G_1 is not finitely presented. If G_1 was finitely presented, then by [Er, Proposition 3.1] there would exist $N \in \mathbb{N}$ with the following property: if $\varphi : H \to G_1$ is a surjective homomorphism from another pro-p group H onto G_1 such that $\operatorname{Ker}\varphi \subseteq \gamma_N H$, then φ is an isomorphism. We see that the latter is impossible by taking $H = \Gamma_0/\langle\{g_i\}_{i=1}^{k}\rangle^{\Gamma_0}$ where k is such that $n_k > N$. □

8.4. Finally, let us say a few words about the current status of the Lubotzky-Sarnak conjecture. Theorem 1.6 shows that the conjecture cannot be established just by using the fact that groups in question are virtually Golod-Shafarevich. Nevertheless, it may be possible to prove the conjecture by purely group-theoretic methods. First of all, Conjecture 1.5 may still hold. Moreover, if Γ is the fundamental group of a compact hyperbolic three-manifold, each finite index subgroup of Γ has a balanced presentation. Using just this property (and the fact that $d_p(H) \geq 5$ for some finite index subgroup H of Γ), Lackenby [La2] showed that $\Gamma_{\widehat{p}}$ (the pro-p completion of Γ) has very large subgroup growth; more precisely, the number of subgroups of $\Gamma_{\widehat{p}}$ of index n is at least $C \cdot 2^{n/(\sqrt{\log n}\log(\log n))}$ for some $C > 0$. Furthermore, using topological techniques, Lackenby [La1] showed that the subgroup growth of $\Gamma_{\widehat{p}}$ is exponential (the largest possible growth type for pro-p groups) provided Γ is arithmetic. As we said before, it is feasible that pro-p completions of groups with (τ) always have smaller subgroup growth.

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