GROUPS GRADED BY ROOT SYSTEMS AND PROPERTY (T)

Mikhail Ershov [1], Andrei Jaikin-Zapirain [2], Martin Kassabov [3] and Zezhou Zhang [4]

[1]University of Virginia, Department of Mathematics, Charlottesville, VA 22904, USA

[2]Departamento de Matemáticas Universidad Autónoma de Madrid and Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Madrid 28049, Spain

[3]Cornell University, Department of Mathematics, Ithaca, NY 14853, USA

[4]University of California San Diego, Department of Mathematics, La Jolla, CA 92093, USA

ABSTRACT. We establish property (T) for a large class of groups graded by root systems, including elementary Chevalley groups and Steinberg groups of rank at least two over finitely generated commutative rings with 1. We also construct a group with property (T) which surjects onto all finite simple groups of Lie type and rank at least two.

1.1. **Introduction.** Groups graded by root systems can be thought of as natural generalizations of Steinberg and Chevalley groups over rings. In recent preprints [5, 18], the authors of this paper determined a sufficient condition which "almost" implies property (T) for a group graded by a root system (see Theorem 1.1 below) and used this result to establish property (T) for Steinberg and Chevalley groups corresponding to reduced irreducible root systems of rank at least two. The goal of this paper is to give an accessible exposition of those results and describe the main ideas used in their proofs.

As will be shown in [5], a substantial part of the general theory of groups graded by root systems can be developed using the term 'root system' in a very broad sense. However, the majority of interesting examples (known to the authors) come from classical root systems (that is, finite crystallographic root systems), so in this paper we will only consider classical root systems (and refer to them simply as 'root systems').

The basic idea behind the definition of a group graded by a root system Φ is that it should be generated by a family of subgroups indexed by Φ, which satisfy commutation relations similar to those between root subgroups of Chevalley and Steinberg groups.

Definition. Let G be a group, Φ a root system and $\{X_\alpha\}_{\alpha\in\Phi}$ a family of subgroups of G. We will say that the groups $\{X_\alpha\}_{\alpha\in\Phi}$ form a Φ-*grading* if for any $\alpha,\beta\in\Phi$ with $\beta\notin\mathbb{R}\alpha$, the following inclusion holds:

$$[X_\alpha, X_\beta]\subseteq\langle X_\gamma : \gamma\in(\mathbb{R}_{\geq 1}\alpha+\mathbb{R}_{\geq 1}\beta)\cap\Phi\rangle. \tag{1.1}$$

If in addition G is generated by the subgroups $\{X_\alpha\}$, we will say that G is *graded by* Φ and that $\{X_\alpha\}_{\alpha\in\Phi}$ is a Φ-grading of G. The groups X_α themselves will be referred to as *root subgroups*.

Clearly, the above definition is too general to yield any interesting structural results, and we are looking for the more restrictive notion of a *strong grading*. A sufficient condition for a Φ-grading to be strong is that the inclusion in (1.1) is an equality; however, requiring equality in general is too restrictive, as, for instance, it fails for Chevalley groups of type B_n over rings where 2 is not invertible. In order to formulate the definition of a strong grading in the general case, we need some additional terminology.

Let Φ be a root system in a Euclidean space V with inner product $(\cdot,\cdot)$. A subset B of Φ will be called *Borel* if B is the set of positive roots with respect to some system of simple roots Π of Φ. Equivalently, B is Borel if there exists $v\in V$, which is not orthogonal to any root in Φ, such that $B=\{\gamma\in\Phi : (\gamma,v)>0\}$.

If B is a Borel subset and Π is the associated set of simple roots, the *boundary* of B, denoted by ∂B, is the set of roots in B which are multiples of roots in Π. In particular, if Φ is reduced, we simply have $\partial B=\Pi$.

Definition. Let Φ be a root system, let G be a group and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading of G. We will say that the grading $\{X_\alpha\}$ is *strong* if for any Borel subset B of Φ, and any root $\gamma \in B \setminus \partial B$ we have $X_\gamma \subseteq \langle X_\beta : \beta \in B \text{ and } \beta \notin \mathbb{R}\gamma\rangle$.

It is easy to see that for any commutative ring R with 1 and any reduced irreducible root system Φ, the elementary Chevalley groups $\mathbb{E}_\Phi(R)$ and the Steinberg group $\mathrm{St}_\Phi(R)$ are strongly graded by Φ.

Our first main theorem asserts that any group strongly graded by an irreducible root system of rank at least two is in some sense close to having property (T).

Theorem 1.1 ([5])**.** *Let Φ be an irreducible root system of rank at least two, and let G be a group which admits a strong Φ-grading $\{X_\alpha\}$. Then the union of root subgroups $\{X_\alpha\}$ is a Kazhdan subset of G (see Definition 1.3).*

By definition, a group G has property (T) if it has a finite Kazhdan subset. Even though the root subgroups are almost never finite, Theorem 1.1 reduces proving property (T) for G to showing that the pair $(G, \cup X_\alpha)$ has relative property (T), and the latter can be achieved in many important examples. In particular, as a consequence of Theorem 1.1 and those results on relative property (T), we obtain the following theorem:

Theorem 1.2 ([5, 18])**.** *Let Φ be a reduced irreducible root system of rank at least two and R a finitely generated ring (with 1). Assume that*

(a) *R is commutative or*
(b) *R is associative and $\Phi = A_n$ (with $n \geq 2$) or*
(c) *R is alternative and $\Phi = A_2$.*

Then the elementary Chevalley group $\mathbb{E}_\Phi(R)$ and the Steinberg group $\mathrm{St}_\Phi(R)$ have property (T).

Remark: Recall that a ring R is called *alternative* if $(xx)y = x(xy)$ and $x(yy) = (xy)y$ for all $x, y \in R$.

Part (b) of Theorem 1.2 was previously established in [4], but the result for root systems of types other than A was only known over rings of Krull dimension one [10]. Later in the paper we will discuss several extensions of Theorem 1.2, dealing with twisted Steinberg groups over rings with involution.

Convention. All rings in this paper are assumed to be unital and all groups are assumed to be discrete.

1.2. **Examples of groups graded by root systems.** In all examples below gradings are strong unless explicitly stated otherwise.

(1) Let Φ be a reduced irreducible root system, let R be a commutative ring, and let $\mathbb{G}_\Phi(R)$ be the corresponding simply-connected Chevalley group. The root subgroups of $\mathbb{G}_\Phi(R)$ with respect to the standard torus clearly form a Φ-grading. Thus the subgroup $\mathbb{E}_\Phi(R)$ of $\mathbb{G}_\Phi(R)$ generated by those root subgroups, which will be referred to as the *elementary Chevalley group*, is graded by Φ.
(2) Let R be an associative ring. The group $\mathrm{EL}_n(R)$ with $n \geq 3$ has a natural A_{n-1}-grading $\{X_{ij}\}_{1\leq i\neq j\leq n}$, where $X_{ij} = \{I_n + rE_{ij} : r \in R\}$. Of course, $\mathrm{EL}_n(R) = \mathbb{E}_{A_{n-1}}(R)$ for R commutative.
(3) Let R be an alternative ring. As shown in [6, Appendix], one can define the A_2-graded group $\mathrm{EL}_3(R)$ via "exponentiation" (in a suitable sense) of the

Lie algebra $\mathfrak{cl}_3(R)$ which, in turn, is defined as the quotient of the Steinberg Lie algebra $\mathfrak{st}_3(R)$ modulo its center.

(4) Let R be an associative ring endowed with an involution $*$, that is, an anti-automorphism of order at most 2. Then $*$ induces the associated involution $A \mapsto A^*$ on the ring $\mathrm{M}_{2n}(R)$ of $2n \times 2n$ matrices over R, where A^* is the transpose of the matrix obtained from A by applying $*$ to each entry. Let $J_{symp} = \sum_{i=1}^{n} E_{i,\bar{i}} - E_{\bar{i},i}$, where $\bar{i} = 2n + 1 - i$. The symplectic group $\mathrm{Sp}_{2n}(R,*)$ is defined by

$$\mathrm{Sp}_{2n}(R) = \{M \in \mathrm{GL}_{2n}(R) : \ M J_{symp} M^* = J_{symp}\}.$$

It can be shown that the following subgroups of $\mathrm{Sp}_{2n}(R)$ form a C_n-grading $(1 \le i, j \le n)$.

$$\begin{aligned} X_{e_i - e_j} &= \{I + rE_{ij} - r^* E_{\bar{j}\bar{i}} : \ r \in R\} \text{ for } i \ne j, \\ X_{e_i + e_j} &= \{I + rE_{i\bar{j}} + r^* E_{j\bar{i}} : \ r \in R\} \text{ for } i < j. \\ X_{-e_i - e_j} &= \{I + rE_{\bar{j}i} + r^* E_{\bar{i}j} : \ r \in R\} \text{ for } i < j, \\ X_{2e_i} &= \{I + rE_{i\bar{i}} : \ r \in R, \ r^* = r\}, \\ X_{-2e_i} &= \{I + rE_{\bar{i}i}. : \ r \in R, \ r^* = r\}. \end{aligned}$$

These root subgroups generate the elementary symplectic group $\mathrm{ESp}_{2n}(R,*)$. Note that $X_\gamma \cong (R,+)$ if γ is a short root, and $X_\gamma \cong (\mathrm{Sym}(R),+)$ if γ is a long root, where $\mathrm{Sym}(R) = \{r \in R : r^* = r\}$.

(5) Again let R be an associative ring with an involution $*$, and let $m \ge 4$ be an integer. Keeping the notations from Example 4, let $J_{unit} = \sum_{i=1}^{m} E_{i,m+1-i}$. The unitary group $U_m(R,*)$ is defined by

$$U_m(R) = \{M \in \mathrm{GL}_m(R) : \ M J_{unit} M^* = J_{unit}\}.$$

If $m = 2k$ is even, the group $\mathrm{U}_{2k}(R,*)$ has a natural C_k-grading, where each short root subgroup is isomorphic to $(R,+)$ and each long root subgroup is isomorphic to $(\mathrm{Asym}(R),+)$ where $\mathrm{Asym}(R) = \{r \in R : r^* = -r\}$.

If $m = 2k+1$ is odd, the group $\mathrm{U}_{2k+1}(R,*)$ has a natural BC_k-grading, where each long root subgroup is isomorphic to $(R,+)$, each double root subgroup is isomorphic to $(\mathrm{Asym}(R),+)$, and short root subgroups have more complicated description (they are nilpotent of class at most 2, typically equal to 2).

In both cases the subgroup of $\mathrm{U}_m(R)$ generated by those root subgroups will be denoted by $\mathrm{EU}_m(R)$. This grading is always strong for $m \ge 6$ and strong for $m = 4, 5$ under some natural condition on the pair $(R,*)$.

(6) For any associative ring R, the group $\mathrm{EL}_n(R)$ actually has an A_k-grading $\{X_{ij}\}$ for any $1 \le k \le n-1$, constructed as follows. Choose integers $a_1, \ldots, a_{k+1} \ge 1$ with $\sum a_i = n$. Thinking of elements of $\mathrm{M}_n(R)$ as $(k+1) \times (k+1)$-block matrices with (i,j)-block having dimensions $a_i \times a_j$, we let X_{ij} be the subgroup generated by all elementary matrices whose nonzero diagonal entry lies in the (i,j)-block.

(7) Given any Φ-grading $\{X_\alpha\}$ of a group G, one can construct another Φ-graded group $\widehat{G}$, which naturally surjects onto G, and called the *graded cover* of G. Somewhat informally, $\widehat{G}$ can be defined as follows. Given any $\alpha, \beta \in \Phi$ with $\beta \notin \mathbb{R}\alpha$, let $\Phi_{\alpha,\beta} = (\mathbb{R}_{\ge 1}\alpha + \mathbb{R}_{\ge 1}\beta) \cap \Phi$. By the commutation relations (1.1), for any $x \in X_\alpha$ and $y \in X_\beta$, there exist

elements $\{z_\gamma(x,y) \in X_\gamma : \gamma \in \Phi_{\alpha,\beta}\}$ such that

$$[x,y] = \prod_{\gamma\in\Phi_{\alpha,\beta}} z_\gamma(x,y) \tag{1.2}$$

(where the product is taken in some fixed order). The group $\widehat{G}$ is the quotient of the free product of the root subgroups $\{X_\alpha\}$ by the commutation relations (1.2). It is clear that $\widehat{G}$ also has a Φ-grading, which is strong whenever the original Φ-grading of G is strong.

If $G = \mathbb{E}_\Phi(R)$, the graded cover of G (with respect to its canonical grading) is the Steinberg group $\mathrm{St}_\Phi(R)$ (this can be taken as the definition of the Steinberg group). Similarly, for any associative ring R, the Steinberg group $\mathrm{St}_n(R)$ is the graded cover of $\mathrm{EL}_n(R)$.

(8) If G is any group graded (resp. strongly graded) by a root system Φ, any quotient of G is also graded (resp. strongly graded) by Φ in an obvious way.

(9) Let R be an alternative ring with involution $*$. Consider $\mathcal{J} := H_3(R,*)$, the space of 3×3 Hermitian matrices over R, and for every $x \in H_3(R,*)$ define the operator $U_x : \mathcal{J} \to \mathcal{J}$ by $U_x(y) = xyx$. Then $\mathcal{J}$ with operators U_x becomes a quadratic Jordan algebra (see [11, Pg 83] for the definition). Note that if $\frac{1}{2} \in R$, then $\mathcal{J}$ is a usual Jordan algebra with product $x \circ y = \frac{1}{2}(xy + yx)$.

For $x, y, z \in \mathcal{J}$, define $V_{x,y}(z) = (U_{x+z} - U_x - U_z)(y)$ and $\mathfrak{L}_0(\mathcal{J}) := \mathrm{span}\{V_{x,y} \ \ x, y \in \mathcal{J}\}$. The celebrated Tits-Kantor-Koecher construction allows us to endow to the abelian group $\mathcal{TKK}(\mathcal{J}) := \mathcal{J}^+ \oplus \mathfrak{L}_0(\mathcal{J}) \oplus \mathcal{J}^-$ with a Lie algebra structure depending only on the U and V operators defined above (see [11, Pg 13]). Such Lie algebra admits a C_3-grading where $\mathcal{J}^\pm$ are the weight spaces corresponding to $\{\pm(e_i + e_j)\}$ and $\mathfrak{L}_0(\mathcal{J})$ contains the zero weight space and the ones corresponding to $\{\pm(e_i - e_j)\}$.

The elementary symplectic group $\mathrm{ESp}_6(R,*)$ is generated by "exponentials" of the nonzero weight subspaces of the above Lie algebra (see [18] and the references therein for details). The group $\mathrm{ESp}_6(R,*)$ is C_3-graded, see [18].

It is an interesting problem to find an abstract characterization of groups in some of the above examples, at least up to graded covers. Such characterization of groups in Examples 1 and 2 of type A_n, $n \ge 3$, D_n and E_n, was obtained in [14]. The ongoing work [19] extends this characterization to groups in Examples 1-4 of types A_n, $n \ge 2$ and C_n, $n \ge 3$ (with additional restrictions for types A_2 and C_3) and may eventually lead to a complete classification of root-graded groups (which can be informally thought of as groups graded by root systems endowed with a suitable action of the Weyl group).

1.3. **Prior work on property (T).**

Definition. Let G be a group and S a subset of G.

(a) The Kazhdan constant $\kappa(G,S)$ is the largest $\varepsilon \ge 0$ with the following property: if V is a unitary representation of G which contains a vector v such that $\|sv - v\| < \varepsilon\|v\|$ for all $s \in S$, then V contains a nonzero G-invariant vector.
(b) S is called a *Kazhdan subset* of G if $\kappa(G,S) > 0$.
(c) G has *property* (T) if it has a finite Kazhdan subset.

If a group G has property (T), then G is finitely generated and moreover any generating subset of G is Kazhdan.

In order to establish property (T) for many interesting groups, one needs the notion of relative property (T). Relative property (T) was first defined for pairs (G, B) where B is a normal subgroup of a group G, but more recently has been extended to the case when B is an arbitrary subset of G.

(a) Let B be a normal subgroup of G. The pair (G, B) has *relative property* (T) if there exists a finite subset S of G such that whenever a unitary representation V of G has a vector $v \in V$ satisfying $\|sv - v\| \leq \varepsilon \|v\|$ for every $s \in S$, there must exist a nonzero B-invariant vector in V.

(b) Assume now that B is an arbitrary subset of G. The pair (G, B) has *relative property* (T) if there exist a finite subset S of G and a function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that if V is any unitary representation of G and $v \in V$ satisfies $\|sv - v\| \leq f(\varepsilon)\|v\|$ for every $s \in S$, then $\|bv - v\| \leq \varepsilon \|v\|$ for every $b \in B$.

The equivalence of definitions (a) and (b) in the case when B is a normal subgroup of G is not trivial, but what one typically uses is the implication "(a)$\Rightarrow$(b)", whose proof is elementary and was first given in [12] (though expressed in a different language there).

Clearly in order to prove that a group G has property (T), it is sufficient to find a subset B of G such that

(i) B is a Kazhdan subset of G, that is, $\kappa(G, B) > 0$

(ii) (G, B) has relative property (T).

Suppose now that $G = \mathrm{EL}_n(R)$ for some $n \geq 3$ and associative ring R. If one wants to prove property (T) using the above general strategy, a natural choice for B is $\mathrm{Elem}_n(R)$, the set of elementary matrices in $\mathrm{EL}_n(R)$ or, equivalently, the union of root subgroups. Verification of condition (ii) in this case is an easy consequence of the following result:

Theorem 1.3. *Let R be a finitely generated associative ring. Then the pair $(\mathrm{EL}_2(R) \ltimes R^2, R^2)$ has relative property (T), where $\mathrm{EL}_2(R)$ acts on R^2 by left multiplication.*

Theorem 1.3 was proved by Burger [2] for $R = \mathbb{Z}$, by Shalom [12] for commutative R, and by Kassabov [8] in the above form. It immediately implies that the pair $(\mathrm{EL}_n(R), \mathrm{Elem}_n(R))$ has relative property (T) (for $n \geq 3$) since for any root subgroup X_{ij} of $\mathrm{EL}_n(R)$ there is a homomorphism $\varphi_{ij} : \mathrm{EL}_2(R) \ltimes R^2 \to \mathrm{EL}_n(R)$ such that $X_{ij} \subset \varphi_{ij}(R^2)$. Shalom [12] also proved that $\mathrm{Elem}_n(R)$ is a Kazdhan subset of $\mathrm{EL}_n(R)$ when R is commutative of Krull dimension 1 (thereby establishing property (T) in this case), using the fact that $\mathrm{EL}_n(R)$ over such rings is known to be boundedly generated by root subgroups.

In later works of Shalom [13] and Vaserstein [17], a generalization of the bounded generation principle was used to prove property (T) for $\mathrm{EL}_n(R)$, $n \geq 3$, over any finitely generated commutative ring R, but that proof followed a more complex strategy and did not involve verification of conditions (i) and (ii) above for some subset B.

In [4], Ershov and Jaikin-Zapirain proved that $\mathrm{Elem}_n(R)$ is a Kazhdan subset of $\mathrm{EL}_n(R)$ (and hence $\mathrm{EL}_n(R)$ has property (T)), but this time by a completely different method, which we describe next.

1.4. **Property (T) for the group associated with a graph of groups.** The main ingredient of the argument in [4] is a criterion for property (T) for groups associated to a graph of groups. In order to describe it we need to introduce a series of definitions.

Definition. Let V be a Hilbert space, and let $\{U_i\}_{i=1}^n$ be subspaces of V, at least one of which is non-trivial. The quantity

$$\operatorname{codist}(\{U_i\}) = \sup\left\{\frac{\|u_1 + \cdots + u_n\|^2}{n(\|u_1\|^2 + \cdots + \|u_n\|^2)} : u_i \in U_i\right\}.$$

will be called the *codistance* between the subspaces $\{U_i\}_{i=1}^n$.

It is clear that $\operatorname{codist}(\{U_i\})$ is a real number in the interval $[\frac{1}{n}, 1]$. Moreover, $\operatorname{codist}(\{U_i\}) = \frac{1}{n}$ if and only if the subspaces $\{U_i\}$ are pairwise orthogonal and $\operatorname{codist}(\{U_i\}) = 1$ whenever the intersection $\cap U_i$ is non-trivial.

It is easy to show that the codistance can also be defined as the square of the cosine of the angle between the subspaces $\operatorname{diag}(V) = \{(v, v, \ldots, v) : v \in V\}$ of V^n and $U_1 \times \ldots \times U_n$:

$$\operatorname{codist}(\{U_i\}) = \sup\left\{\frac{\|\sum_{i=1}^n \langle v, u_i\rangle\|^2}{\|v\|^2(\sum_{i=1}^n \|u_i\|^2)} : u_i \in U_i, v \in V\right\}.$$

An important consequence of the latter formula is that $\operatorname{codist}(\{U_i\}) < 1$ if and only if a unit vector $v \in V$ cannot be arbitrarily close to U_i for all i.

Definition. Let $\{H_i\}_{i=1}^n$ be subgroups of the same group, and let $G = \langle H_1, \ldots, H_n\rangle$. The *codistance* between $\{H_i\}$, denoted by $\operatorname{codist}(\{H_i\})$, is defined to be the supremum of the quantities $\operatorname{codist}(V^{H_1}, \ldots, V^{H_n})$, where V ranges over all unitary representations of G without nonzero G-invariant vectors and V^{H_i} denotes the subspace of H_i-invariant vectors.

Proposition 1.4 ([4]). *Suppose that* $G = \langle H_1, \ldots, H_n\rangle$. *Then*

(a) $\bigcup H_i$ *is a Kazhdan subset of* $G \Leftrightarrow \operatorname{codist}(\{H_i\}) < 1$;

(b) *If* $\operatorname{codist}(\{H_i\}) < 1$ *and* S_i *is a Kazhdan subset of* H_i, *then* $\cup S_i$ *is a Kazhdan subset of* G.

There are very few examples where one can prove that $\operatorname{codist}(\{H_i\}) < 1$ by directly analyzing representations of the group $G = \langle H_1, \ldots, H_n\rangle$. However, in many cases one can show that $\operatorname{codist}(\{H_i\}_{i=1}^n) < 1$ by estimating codistances between suitable subsets of the set $\{H_1, \ldots, H_n\}$ (combined with some additional information on the subgroups $\{H_i\}$). The first result of this kind was obtained by Dymara and Januszkiewicz in [3] who showed that $\operatorname{codist}(\{H_i\}) < 1$ whenever for any two indices $k \neq l$, the codistance $\operatorname{codist}(H_k, H_l)$ is sufficiently close to $\frac{1}{2}$; a quantitative improvement of this result [4, Thm 1.2] asserts that requiring $\operatorname{codist}(H_k, H_l) < \frac{n}{2(n-1)}$ is sufficient. Moreover, it was shown in [4] that a similar principle can be applied in a more complex setting when we are given a graph of groups decomposition of the group G over certain finite graph, as defined below.

Graph-theoretic conventions. All graphs we consider are assumed non-oriented and without loops. The sets of vertices and edges of a graph Γ will be denoted by $\mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma)$, respectively. Given two vertices ν and ν', we will write $\nu \sim \nu'$ if they are connected by an edge in Γ; similarly, for a vertex ν and an edge e we will write $\nu \sim e$ if ν is an endpoint of e.

If Γ is regular, by $\Delta(\Gamma)$ we denote its Laplacian (our convention is that the matrix of Δ is $kI - A$ where k is the degree of Γ and A is the adjacency matrix). Finally, by $\lambda_1(\Delta)$ we will denote the smallest nonzero eigenvalue of Δ.

Definition. Let G be a group and Γ a graph. A *graph of groups decomposition (or just a decomposition) of* G *over* Γ is a choice of a vertex subgroup $G_\nu \subseteq G$ for every $\nu \in \mathcal{V}(\Gamma)$ and an edge subgroup $G_e \subseteq G$ for every $e \in \mathcal{E}(\Gamma)$ such that

(a) The vertex subgroups $\{G_\nu : \nu \in \mathcal{V}(\Gamma)\}$ generate G;
(b) If a vertex ν is an endpoint of an edge e, then $G_e \subseteq G_\nu$
(c) For each $\nu \in \mathcal{V}(\Gamma)$ the vertex subgroup G_ν is generated by the edge subgroups $\{G_e : \nu \sim e\}$

The following criterion from [4] informally says that if the codistance between the edge subgroups at each vertex is small and the graph Γ is highly connected (i.e. $\lambda_1(\Delta)$ is large), then the union of the edge subgroups is a Kazhdan subset.

Theorem 1.5 ([4, Thm 5.1]). *Let Γ be a finite connected k-regular graph and let G be a group with a given decomposition over Γ. For each $\nu \in \mathcal{V}(\Gamma)$ let p_ν be the codistance between the subgroups $\{G_e : \nu \sim e\}$ of G_ν, and let $p = \max_\nu p_\nu$. If $p < \frac{\lambda_1(\Delta)}{2k}$, then $\mathrm{codist}(\{G_\nu\}_{\nu\in\mathcal{V}(\Gamma)}) < 1$, and therefore $\cup_{\nu\in\mathcal{V}(\Gamma)} G_\nu$ is a Kazhdan subset of G. Moreover, $\cup_{e\in\mathcal{E}(\Gamma)} G_e$ is also a Kazhdan subset of G.*

Remark: Since $\lambda_1(\Delta) \le 2k$, the assumption $p < \frac{\lambda_1(\Delta)}{2k}$ implies that each $p_\nu < 1$, so the last assertion of Theorem 1.5 follows from Proposition 1.4.

Example 1.6. Suppose that a group G is generated by three subgroups H_1, H_2, H_3. Let Γ be a complete graph on three vertices labeled v_{12}, v_{13} and v_{23}, and denote the edge between v_{ij} and v_{ik} by e_i (with i, j, k distinct). Define the vertex and edge subgroups by $G_{v_{ij}} = \langle H_i, H_j\rangle$ and $G_{e_i} = \langle H_i\rangle$. Then $p = \max\{\mathrm{codist}(H_i, H_j)\}$, $k = 2$ and $\lambda_1(\Delta) = 3$. By Theorem 1.5, $\kappa(G, H_1 \cup H_2 \cup H_3) > 0$ whenever $p < \frac{3}{4}$, so in this way we recover [4, Thm 1.2] mentioned above in the case $n = 3$.

1.5. **Proof of property (T) for $\mathrm{EL}_n(R)$, with R arbitrary.** The following key result from [4] asserts that for any group G strongly graded by A_2, the union of root subgroups is a Kazhdan subset of G. To give the reader a better feel, we formulate this result explicitly, unwinding the definition of a strong A_2-grading.

Proposition 1.7. *Let G be a group generated by 6 subgroups $\{X_{ij} : 1 \le i, j \le 3,\ i \ne j\}$ such that for any permutation i, j, k of the set $\{1, 2, 3\}$ the following conditions hold:*

(a) X_{ij} is abelian; *(b) X_{ij} and X_{ik} commute;*
(c) X_{ji} and X_{ki} commute; *(d) $[X_{ij}, X_{jk}] = X_{ik}$.*

Then $\kappa(G, \bigcup X_{ij}) \ge \frac{1}{8}$.

The group G satisfying the above hypotheses has a natural decomposition over the graph with six vertices indexed by pairs (i, j) where $1 \le i \ne j \le 3$, in which (i, j) is connected to every other vertex except (j, i). The vertex subgroup at the vertex (i, j) is $G_{ij} = \langle X_{ik}, X_{kj}\rangle$, where $k \in \{1, 2, 3\}$ is distinct from both i and j. If e is the edge joining (i, j) and (i, k) (with i, j, k distinct), the edge subgroup G_e is $X_{ij}X_{ik}$, and if e joins (i, j) and (k, i), we set $G_e = X_{kj}$. An easy computation shows that the codistance between edge subgroups at each vertex is bounded above by $1/2$ (typically equal to $1/2$), while the quantity $\frac{\lambda_1(\Delta)}{2k}$ is also equal to $1/2$.

Thus, Theorem 1.5 is not directly applicable in this situation and cannot be used to prove Proposition 1.7. Instead the latter is proved in [4] by adapting the proof of Theorem 1.5 and using conditions (a)-(d) more efficiently. In this paper we will introduce a generalized spectral criterion (Theorem 1.8), which can be used to prove not only Proposition 1.7, but its generalization Theorem 1.1.

We finish by explaining how to deduce property (T) for $\mathrm{EL}_n(R)$, $n \ge 3$, from Proposition 1.7. Recall that by Theorem 1.3 we only need to show that the union of root subgroups of $\mathrm{EL}_n(R)$ is a Kazhdan subset. If $n = 3$, this follows directly from Proposition 1.7. If $n > 3$, we apply Proposition 1.7 to the A_2-grading of $\mathrm{EL}_n(R)$ described in the Example 6 of § 2.1, and use the fact that each root subgroup in that A_2-grading is a bounded product of usual root subgroups of $\mathrm{EL}_n(R)$.

1.6. **Generalized spectral criterion.** Before stating the generalized spectral criterion, we give a brief outline of the proof of the basic spectral criterion (Theorem 1.5), which will also motivate the statement of the former.

So, let G be a group with a chosen decomposition over a finite k-regular graph Γ. Let V be a unitary representation of G without invariant vectors, and let Ω be the Hilbert space of all functions $f : \mathcal{V}(\Gamma) \to V$, with the inner product $\langle f, g\rangle = \sum_{\nu\in\mathcal{V}(\Gamma)}\langle f(\nu), g(\nu)\rangle$. Let $U \subseteq \Omega$ be the subspace of all constant functions and $W \subseteq \Omega$ be the subspace of all functions f such that $f(\nu) \in V^{G_\nu}$ for each $\nu \in \mathcal{V}(\Gamma)$. According to Proposition 1.4, in order to prove that $\cup_{\nu\in\mathcal{V}(\Gamma)}G_\nu$ is a Kazhdan subset of G, it suffices to show that

$$\mathrm{codist}(U, W) \le 1 - \varepsilon \text{ for some } \varepsilon > 0, \text{ independent of } V. \tag{1.3}$$

Let V' be the closure of $U + W$. Since V has no G-invariant vectors, $U \cap W = \{0\}$, which implies that

$$\mathrm{codist}(U, W) = \mathrm{codist}(U^\perp, W^\perp), \tag{1.4}$$

where $U^\perp$ and $W^\perp$ are the orthogonal complements in V' of U and W, respectively. The Laplacian operator Δ of Γ naturally acts on the space Ω by $(\Delta f)(y) = \sum_{z\sim y}(f(y) - f(z))$, and it is easy to show that $U^\perp = P_{V'}\Delta(W)$, where P_Z denotes the orthogonal projection onto a subspace Z. Combining this observation with (1.4), we reduce the desired inequality (1.3) to the following condition on the Laplacian:

For any $f \in W$, the element Δf cannot be almost orthogonal to W, that is,

$$\|P_W(\Delta f)\| \ge \varepsilon\|\Delta f\| \text{ for some } \varepsilon > 0 \text{ independent of } V \text{ and } f. \tag{1.5}$$

In order to establish (1.5), we consider the decomposition $V = W \oplus W^\perp$, and write any $f \in V$ as the sum of its projections onto W and $W^\perp$: $f = P_W(f) + P_{W^\perp}(f)$. It is not difficult to show that

$$\frac{\lambda_1(\Delta)}{2k}\|P_W(\Delta f)\|^2 + \frac{\lambda_1(\Delta)}{2kp}\|P_{W^\perp}(\Delta f)\|^2 \le \|\Delta f\|^2 = \|P_W(\Delta f)\|^2 + \|P_{W^\perp}\Delta f\|^2.$$

Thus, if $p < \frac{\lambda_1(\Delta)}{2k}$ (which is an assumption in Theorem 1.5), the above inequality implies that $\|P_W(\Delta f)\|$ cannot be too small relative to $\|\Delta f\|$, thus finishing the proof.

Keeping the notations from Theorem 1.5, assume now that we are in the boundary case $p = \frac{\lambda_1(\Delta)}{2k}$. Also suppose that for each vertex ν there is a normal subgroup CG_ν of G_ν such that for any representation V_ν of G_ν without CG_ν-invariant vectors, the codistance between the subspaces $\{V_\nu^{G_e} : e \sim \nu\}$ is bounded above by $p(1-\delta)$ for some absolute $\delta > 0$.

To make use of this condition, we now decompose V into a direct sum of three subspaces $V = W_1 \oplus W_2 \oplus W_3$, where

$$W_1 = W, \quad W_3 = \{f \in \Omega : f(\nu) \in V^{CG_\nu}\}, \quad W_2 = (W_1 \oplus W_3)^\perp.$$

Denoting by P_i the orthogonal projection onto W_i and repeating the above argument, for any $f \in W$ we get

$$\frac{\lambda_1(\Delta)}{2k}\|P_1(\Delta f)\|^2 + \frac{\lambda_1(\Delta)}{2kp}\|P_2(\Delta f)\|^2 + \frac{\lambda_1(\Delta)}{2kp(1-\delta)}\|P_3(\Delta f)\|^2 \\ \leq \|\Delta f\|^2 = \|P_1(\Delta f)\|^2 + \|P_2(\Delta f)\|^2 + \|P_3(\Delta f)\|^2.$$

By our assumption the coefficient of $\|P_2(\Delta f)\|^2$ on the left-hand side is equal to 1 and the coefficient of $\|P_3(\Delta f)\|^2$ is larger than 1. Thus, the inequality implies that if the ratio $\|P_1(\Delta f)\|/\|\Delta f\|$ is close to 0, then the ratio $\|P_2(\Delta f)\|/\|\Delta f\|$ must be close to 1. Explicitly disallowing the latter possibility, we obtain the following generalized version of the spectral criterion.

Theorem 1.8 ([5]). *Let Γ be a finite connected k-regular graph. Let G be a group with a chosen decomposition over Γ, and for each $\nu \in \mathcal{V}(\Gamma)$ choose a normal subgroup CG_ν of G_ν, called the core subgroup. Let $p = \frac{\lambda_1(\Delta)}{2k}$, where Δ is the Laplacian of Γ. Suppose that*

- (i) *For each $\nu \in \mathcal{V}(\Gamma)$, the codistance between the edge subgroups $\{G_e : \nu \sim e\}$ of G_ν is bounded above by p.*
- (ii) *There exists $\delta > 0$ such that for any $\nu \in \mathcal{V}(\Gamma)$ and any unitary representation V of the vertex group G_ν without CG_ν-invariant vectors, the codistance between the fixed subspaces of G_e, with $\nu \sim e$, is bounded above by $p(1-\delta)$;*
- (iii) *There exists $\delta' > 0$ such that $\|P_2(\Delta f)\| < (1-\delta')\|\Delta f\|$ for any $f \in W$, where P_2 is defined as above.*

Then $\cup_{e \in \mathcal{E}(\Gamma)} G_e$ is a Kazhdan subset of G.

1.7. **Sketch of the proof of Theorem 1.1.** Let G be a group with a strong Φ-grading $\{X_\alpha\}_{\alpha\in\Phi}$ for some irreducible root system Φ of rank at least two. The first key observation is that G has a natural decomposition over certain graph $\Gamma_l = \Gamma_l(\Phi)$, which we call the large Weyl graph of Φ. The reader can easily see that the decomposition of A_2-graded groups introduced in § 1.5 is a special case of the following construction.

The vertices of $\Gamma_l(\Phi)$ are labeled by the Borel subsets of Φ, and two distinct vertices B and B' are connected if and only if $B \cap B' \neq \emptyset$ (equivalently, $B' \neq -B$). For a subset S of B, we put $G_S = \langle X_\alpha : \alpha \in S\rangle$. Then the vertex subgroup at a vertex B is defined to be G_B, and if e is an edge joining vertices B and B', we define the edge subgroup at e to be $G_{B\cap B'}$. Finally, the core subgroup at a vertex B is set to be $G_{B\setminus\partial B}$, that is, $CG_B = G_{B\setminus\partial B}$ in the notations of Theorem 1.8.

We claim that Theorem 1.8 is applicable to this decomposition of G over Γ_l. Below we outline verifications of conditions (i) and (ii) of Theorem 1.8, skipping the more technical argument needed for part (iii). The proof is based on the following result about codistances of certain families of subgroups in nilpotent groups:

Lemma 1.9. *Let N be a nilpotent group, and let $\{X_i\}_{i=1}^n$ be a finite family of subgroups of N such that for each $1 \leq i \leq n$, the product set $N_i = \prod_{j=i}^n X_j$ is a normal subgroup of N, $N_1 = N$ and $[N_i, N] \subseteq N_{i+1}$ for each i. Let $\{G_j\}_{j=1}^m$ be another family of subgroups of N, and let $l \in \mathbb{Z}$ be such that for each $1 \leq i \leq n$, the subgroup X_i lies in G_j for at least l distinct values of j. Then* $\mathrm{codist}(G_1, \ldots, G_m) \leq \frac{m-l}{m}$.

Let us now go back to the proof of Theorem 1.1. Denote the Laplacian of Γ_l by Δ, let $k = \deg(\Gamma_l)$ and let $d = |\mathcal{V}(\Gamma_l)|$. It is easy to see that $\lambda_1(\Delta) = k = d - 2$, so the ratio $p = \frac{\lambda_1(\Delta)}{2k}$ is equal to 1/2.

Now fix a Borel set B and let $N = G_B$. If we let $\{X_i\}_{i=1}^n$ be suitably ordered root subgroups contained in N, the first hypothesis of Lemma 1.9 clearly holds. Letting $\{G_j\}$ be the edge subgroups corresponding to edges incident to B, we get that in the notations of Lemma 1.9, $m = \deg(\Gamma_l)$ and $l = m/2$. The latter holds since B is connected by an edge with all other Borels except its opposite $-B$. Those Borels split into pairs of mutually opposite ones, and each root subgroup X_i lies in precisely one Borel from each pair. Thus, the ratio $\frac{m-l}{m}$ is equal to 1/2, so condition (i) of Theorem 1.8 follows from Lemma 1.9.

The proof of condition (ii) follows from a more technical version of Lemma 1.9, which we do not state here. We only mention that it is proved by essentially the same computation as Lemma 1.9 combined with the following general fact:

Proposition 1.10 ([5])**.** *If a nilpotent group N of class c is generated by subgroups $X_1, \ldots, X_k$, then* $\operatorname{codist}(X_1, \ldots, X_k) \leq 1 - \varepsilon$ *where* $\varepsilon = (k \cdot 4^{c-1})^{-1}$.

Applying Theorem 1.8, we get that the union of the vertex subgroups G_B is a Kazhdan subset of G. However, each G_B is a bounded product of root subgroups X_α, which implies that $\cup X_\alpha$ is also a Kazhdan subset, thus finishing the proof.

1.8. **Property** (T) **for Steinberg groups.** We start by deducing Theorem 1.2 from Theorem 1.1. Since elementary Chevalley groups are quotients of the associated Steinberg groups, we only need to prove property (T) for Steinberg groups.

Let Φ be a reduced irreducible root system of rank at least two, R a finitely generated commutative ring and $\{X_\alpha\}_{\alpha\in\Phi}$ the root subgroups of the Steinberg group $\mathrm{St}_\Phi(R)$. By Theorem 1.1, $\cup_{\alpha\in\Phi} X_\alpha$ is a Kazhdan subset of $\mathrm{St}_\Phi(R)$, so we only need to prove relative property (T) for the pair $(\mathrm{St}_\Phi(R), \cup_{\alpha\in\Phi} X_\alpha)$.

We shall use the following generalization of Theorem 1.3 where we replace the group $\mathrm{EL}_2(R)$ by $R * R$ and allow R to be alternative.

Theorem 1.11 ([18])**.** *Let R be a finitely generated alternative ring, and denote by $R * R$ the free product of two copies of the additive group of R. Then the pair $((R*R) \ltimes R^2, R^2)$ has relative property (T), where the first copy of R in $R*R$ acts on R^2 by upper-triangular matrices and the second copy of R acts by lower-triangular matrices.*

The key fact which enables us to prove this generalization is that for any finitely generated alternative ring R, its ring of left multiplication operators $L(R)$, an associative ring, is finitely generated.

If Φ is simply-laced, relative property (T) for the pair $(\mathrm{St}_\Phi(R), \cup_{\alpha\in\Phi} X_\alpha)$ follows from Theorem 1.11 by the same argument as in the case of $\mathrm{EL}_n(R)$ discussed above. If Φ is not simply-laced, we use a similar argument to reduce relative property (T) for $(\mathrm{St}_\Phi(R), \cup_{\alpha\in\Phi} X_\alpha)$ to relative property (T) for certain semi-direct product $(Q \ltimes N, N)$ where N is nilpotent of class at most 3 (at most 2 if $\Phi \neq G_2$). The latter is proved by combining Theorem 1.11 with the following result:

Theorem 1.12 ([5])**.** *Let G be a group, N a normal subgroup of G and Z a subgroup of $Z(G) \cap N$. Assume that the following properties hold*

(1) *$(G/Z, N/Z)$ has relative property (T),*

(2) *G/N is finitely generated,*

(3) $|Z : Z \cap [N, G]|$ *is finite.*
Then (G, N) *has relative property* (T).

Next we discuss property (T) for elementary symplectic groups and elementary unitary groups in odd dimensions (introduced in Examples 4, 5 and 9), as well as their graded covers, for which we shall introduce special notations.

Definition. Let R be a ring with an involution $*$, and let $n \geq 2$ be an integer. If $n = 3$, assume that R is alternative, and if $n \neq 3$, assume that R is associative.

(i) The graded cover of the elementary symplectic group $\mathrm{ESp}_{2n}(R, *)$ (with respect to its canonical grading) will be denoted by $\mathrm{St}^{-1}_{C_n}(R, *)$.
(ii) The graded cover of the elementary unitary group $\mathrm{EU}_{2n}(R, *)$ will be denoted by $\mathrm{St}^{1}_{C_n}(R, *)$.

The reason we are using very similar notations for these groups is that one can actually define the whole family of twisted Steinberg groups $\mathrm{St}^{\omega}_{C_n}(R, *)$ where ω is any central element of R satisfying $\omega^* \omega = 1$. The subscript C_n simply indicates that those groups are C_n-graded.

Theorem 1.13 ([5, 18])**.** *Let R be a finitely generated ring with involution $*$ and $n \geq 3$. If $n = 3$, assume that R is alternative and $\omega = -1$, and if $n \neq 3$, assume that R is associative and $\omega = \pm 1$. Let $J = \{r \in R : \ r^* = -\omega r\}$ and assume that there exist $a_1, \ldots, a_k \in J$ such that every element $a \in J$ can be expressed as $a = \sum_{i=1}^k s_i a_i s_i^* + (r - r^* \omega)$ with $s_i, r \in R$. Then the following hold:*

(a) *The group $\mathrm{St}^{\omega}_{C_n}(R, *)$ has property (T).*
(b) *Assume in addition that $\omega = -1$ and R is a finitely generated right module over its subring generated by a finite subset of J. Then the group $\mathrm{St}^{-1}_{C_2}(R, *)$ has property (T).*

Remark: The group $\mathrm{ESp}_{2n}(R, *)$ (resp. $\mathrm{EU}_{2n}(R, *)$) has property (T) whenever $\mathrm{St}^{-1}_{C_n}(R, *)$ (resp. $\mathrm{St}^{1}_{C_n}(R, *)$) has property (T).

The method of proof of Theorem 1.13 is similar to that of Theorem 1.2, although verifying that the grading is strong and establishing relative property (T) for suitable pairs is computationally more involved, which is indicated by the somewhat technical assumptions on the pair $(R, *)$ in the above theorem.

In [5], we will establish analogues of Theorem 1.13 dealing with other types of twisted elementary Chevalley groups of rank at least two and their graded covers. These include

(i) twisted groups of type ${}^2A_{2n+1}$ over associative rings with involutions (see Example 5),
(ii) twisted groups of types ${}^2D_n$, $n \geq 4$, and ${}^2E_6$ over commutative rings with involution,
(iii) twisted groups of type ${}^3D_4$ over commutative rings endowed with an automorphism of order 3, and
(iv) twisted groups of type ${}^2F_4$ which can be defined over a commutative ring R of characteristic 2 with a monomorphism $* : R \to R$ such that $(r^*)^* = r^2$ for all $r \in R$.

The groups in family (iv) are graded by non-crystallographic systems, so to prove property (T) for these groups one needs the version of Theorem 1.1 dealing with general root systems (that is, root systems in the sense of [5]). We also note that groups of type ${}^2F_4$ are known as Ree groups in the case when R is a finite field and as Tits groups in the case when R is an arbitrary field (introduced in [16]), but we are not aware of any previous works on these groups over non-fields.

1.9. **Application to expanders.** In 1973, Margulis observed that for any group G with property (T), the Cayley graphs of finite quotients of G form a family of expander graphs; in fact, this gave the first explicit construction of expanders. Since then a variety of methods for producing expander families have been developed, and many different families of finite groups were shown to be expanding – formally, a family of finite groups is called a *family of expanders* if the Cayley graphs of those groups with respect to some generating sets of uniformly bounded size form a family of expanders. One of the main theorems in this area asserts that all (non-abelian) finite simple groups form a family of expanders. The proof of this result is spread over several papers [8, 9, 1].

Given a family $\mathcal{F}$ of finite groups, a group G which surjects onto every group in $\mathcal{F}$ will be called a *mother group* for $\mathcal{F}$. Clearly, a family $\mathcal{F}$ admits a finitely generated mother group if and only if the (minimal) number of generators of groups in $\mathcal{F}$ is uniformly bounded. If in addition $\mathcal{F}$ is an expanding family, one may ask if it admits a mother group with property (T). In particular, it is interesting to determine which families of finite simple groups have a mother group with (T). We shall obtain a positive answer to this question for "most" finite simple groups of Lie type.

Theorem 1.14 ([5])**.** *The family of all finite simple groups of Lie type and rank at least two has a mother group with property* (T).

Theorem 1.14 cannot be extended to all finite simple groups (even those of Lie type) since it is well known that the family $\{\mathrm{SL}_2(\mathbb{F}_p)\}$ does not have a mother group with (T). However, it is still possible that the family of all finite simple groups has a mother group with property (τ) (certain weaker version of property (T)), which would be sufficient for expansion.

To prove Theorem 1.14 we first divide all finite simple groups of Lie type and rank at least two into finitely many subfamilies. Then for each subfamily $\mathcal{F}$ we construct a strong Ψ-grading for each group $G \in \mathcal{F}$ by a suitable root system Ψ (depending only on $\mathcal{F}$). Finally we show that all groups in $\mathcal{F}$ are quotients of a (possibly twisted) Steinberg group associated to Ψ, which can be shown to have property (T) by methods described in this paper. The precise realization of this strategy is quite involved, so we will illustrate it by a series of examples, omitting the more technical cases.

(1) Let Φ be a reduced irreducible root system of rank at least two. Then the simple group of Lie type Φ over a finite field F is a quotient of $\mathrm{St}_\Phi(F)$ and so it is a quotient of $\mathrm{St}_\Phi(\mathbb{Z}[t])$. The latter group has property (T) by Theorem 1.2.

(2) Let $n \geq 2$. The simple groups $\mathrm{PSU}_{2n}(\mathbb{F}_q)$ (which are twisted Lie groups of type ${}^2A_{2n-1}$) are quotients of $\mathrm{St}^1_{C_n}(\mathbb{F}_{q^2}, *)$ where $*$ is the (unique) automorphism of $\mathbb{F}_{q^2}$ of order 2. It is easy to show that the groups $\mathrm{St}^1_{C_n}(\mathbb{F}_{q^2}, *)$ and $\mathrm{St}^{-1}_{C_n}(\mathbb{F}_{q^2}, *)$ are isomorphic.

    Let $R = \mathbb{Z}[t_1, t_2]$, the ring of polynomials in two (commuting) variables, and let $* : R \to R$ be the involution which swaps t_1 and t_2. Then $\mathrm{St}^{-1}_{C_n}(R, *)$ satisfies the hypotheses of Theorem 1.13 and all the groups $\mathrm{St}^{-1}_{C_n}(\mathbb{F}_{q^2}, *)$ are quotients of $\mathrm{St}^{-1}_{C_n}(R, *)$.

(3) The simple groups $\mathrm{PSL}_{3k}(F)$, F a finite field, are quotients of $\mathrm{EL}_3(\mathbb{Z}\langle x,y\rangle)$. The latter group has property (T) by [4].

(4) Let $n \geq 1$, let R be the free associative ring $\mathbb{Z}\langle x,y,z\rangle$ and let $*$ be the involution of R that fixes x,y,z. Then we can realize $\mathrm{PSp}_{6n}(F)$, F a finite field, as a quotient of $\mathrm{St}_{C_3}^{-1}(R,*)$. First observe that $\mathrm{PSp}_{6n}(F)$ is a quotient of $\mathrm{St}_{C_3}^{-1}(\mathrm{M}_n(F),*)$, where $*$ is the transposition.

The ring $\mathrm{M}_n(F)$ can be generated by two symmetric matrices, and there is a surjection $\mathrm{St}_{C_3}^{-1}(R,*) \to \mathrm{St}_{C_3}^{-1}(\mathrm{M}_n(F),*)$ which sends x and y to those matrices and z to E_{11}. By Theorem 1.13, $\mathrm{St}_{C_3}^{-1}(R,*)$ has property (T).

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References

[1] E. Breuillard, B. Green and T. Tao, Suzuki groups as expanders, *Groups Geom. Dyn.* 5 (2011), no. 2, 281–299.

[2] M. Burger, Kazhdan constants for $SL(3,\mathbb{Z})$, *J. Reine Angew. Math.* 413 (1991), 36–67.

[3] J. Dymara and T. Januszkiewicz, Cohomology of buildings and their automorphism groups, *Invent. Math.* 150 (2002), no. 3, 579–627.

[4] M. Ershov and A. Jaikin-Zapirain, Property (T) for noncommutative universal lattices, *Invent. Math.* 179 (2010), no. 2, 303–347.

[5] M. Ershov, A. Jaikin-Zapirain and M. Kassabov, Property (T) for groups graded by root systems, preprint (2014), arXiv:1102.0031.v2.

[6] J. Faulkner, Barbilian planes, *Geom. Dedicata* 30 (1989), no. 2, 125-181.

[7] M. Kassabov, Universal lattices and unbounded rank expanders, *Invent. Math.* 170 (2007), no. 2, 297–326.

[8] M. Kassabov, Symmetric groups and expander graphs. *Invent. Math.* 170 (2007), no. 2, 327-354.

[9] M. Kassabov, A. Lubotzky and N. Nikolov, Finite simple groups as expanders, *Proc. Natl. Acad. Sci. USA* 103 (2006), no. 16, 6116–6119.

[10] D. Kazhdan, On the connection of the dual space of a group with the structure of its closed subgroups. (Russian) *Funkcional. Anal. i Priložen.* 1 (1967), 71–74.

[11] K. McCrimmon, A general theory of Jordan rings, *Proc. Nat. Acad. Sci. U.S.A.* 56 (1966), 1072–1079.

[12] Y. Shalom, Bounded generation and Kazhdan's property (T), *Inst. Hautes Études Sci. Publ. Math.* No. 90 (1999), 145–168.

[13] Y. Shalom, The algebraization of Kazhdan's property (T), *International Congress of Mathematicians.* Vol. II, 1283–1310, Eur. Math. Soc., Zürich, 2006.

[14] Z. Shi, Groups graded by finite root systems. *Tohoku Math. J. (2)* 45 (1993), no. 1, 89-108.

[15] L. Vaserstein, Groups having the property T. (Russian) *Funkcional. Anal. i Prilozen.* 2 (1968), no. 2, 86.

[16] J. Tits, Moufang octagons and the Ree groups of type ${}^2F_4$. *Amer. J. Math.* 105 (1983), no. 2, 539–594.

[17] L. Vaserstein, Bounded reduction of invertible matrices over polynomial rings by addition operations, preprint (2007).

[18] Z. Zhang, Nonassociative algebras and groups with property (T), preprint (2014), arXiv:1311.4089v2, to appear in Int. J. Alg. Comp.

[19] Z. Zhang, work in preparation