

KAZHDAN GROUPS WHOSE FC-RADICAL IS NOT VIRTUALLY ABELIAN

MIKHAIL ERSHOV

ABSTRACT. We construct examples of residually finite groups with Kazhdan's property (T) whose FC-radical is not virtually abelian. This answers a question of Popa and Vaes about possible fundamental groups of II_1 factors arising from Kazhdan groups.

1. INTRODUCTION

If G is a group, the *FC-radical* of G , denoted by $FC(G)$, is the set of all elements of G which centralize a finite index subgroup of G . Equivalently, $FC(G)$ is the set of all elements of G with finite conjugacy class.

In [4] Popa and Vaes asked the following question:

Question 1.1. *Does there exist a residually finite (discrete) group with Kazhdan's property (T) whose FC-radical is not virtually abelian?*

This question was motivated by [4, Theorem 6.4(b)], which asserts that any group satisfying these conditions admits a free ergodic profinite action whose associated II_1 factor has all positive real numbers in its fundamental group. No Kazhdan group with the latter property was previously known.

In this short note we give a positive answer to Question 1.1 using Golod-Shafarevich groups.

We shall prove the following theorem:

Theorem 1.2. *Every Golod-Shafarevich group has a residually finite quotient whose FC-radical is not virtually abelian.*

In [2] it was shown that there exist Golod-Shafarevich groups with property (T). Since property (T) is preserved by quotients, applying Theorem 1.2 to any Golod-Shafarevich group with (T), we obtain a group which settles Question 1.1.

Acknowledgments. I am grateful to Mark Sapir for telling me about Question 1.1.

2. CONSTRUCTION

Informally speaking, a finitely generated group G is Golod-Shafarevich if it has a presentation with a “small” set of relators, where relators are counted with suitable weights. The formal definition is given below.

Definition. Let G be a finitely generated group. Given a prime p let $\mathbb{F}_p[G]$ be the $\mathbb{F}_p$ -group algebra of G and I the augmentation ideal of $\mathbb{F}_p[G]$. Let $\{\omega_n G\}_{n \geq 1}$ be the Zassenhaus

2010 *Mathematics Subject Classification.* Primary 20F05, 20F24, Secondary 20E18.

Key words and phrases. Kazhdan's property (T), Golod-Shafarevich groups, FC-radical.

The author was supported in part by the NSF grants DMS-0901703 and DMS-1201452 and the Sloan Research Fellowship grant BR 2011-105.

p -filtration of G defined by $\omega_n G = \{g \in G : g - 1 \in I^n\}$. For each $g \in G \setminus \bigcap_{n \in \mathbb{N}} \omega_n G$ we put $\deg_p(g)$ to be the largest n such that $g \in \omega_n G$.

We will need the following well-known properties of the Zassenhaus filtrations. They have been originally established by Jennings [3]; see also [1, Ch.11,12]:

- (A) The subgroups $\{\omega_n G\}$ are of finite index in G . Moreover, they form a base for the pro- p topology on G , and thus $\bigcap_{n \in \mathbb{N}} \omega_n G$ is the kernel of the natural map from G to its pro- p completion $G_{\widehat{p}}$. In particular, if G is residually- p , then $\bigcap_{n \in \mathbb{N}} \omega_n G = \{1\}$, so $\deg_p(g)$ is defined for any $g \in G \setminus \{1\}$.
- (B) $\omega_n G = \prod_{p^i \cdot j \geq n} (\gamma_j G)^{p^i}$ where $\gamma_j G$ is the j^{th} term of the lower central series of G and H^m is the subgroup generated by m^{th} powers of a group H . Therefore, if $\varphi : G \rightarrow K$ is a group homomorphism, then $\omega_n(\varphi(G)) = \varphi(\omega_n G)$.

Definition. Fix a prime number p .

- (a) A group presentation $\langle X | R \rangle$, with X finite, is said to satisfy the *Golod-Shafarevich* (GS) condition (with respect to p), if there is a real number $t \in (0, 1)$ such that

$$1 - H_X(t) + H_R(t) < 0 \text{ where } H_X(t) = |X|t \text{ and } H_R(t) = \sum_{r \in R} t^{\deg_p(r)}.$$

- (b) A group G is called *Golod-Shafarevich* if it has a presentation satisfying the Golod-Shafarevich condition.

Any Golod-Shafarevich group G is infinite. In fact, its pro- p completion $G_{\widehat{p}}$ must be infinite and moreover satisfies a number of largeness properties (see, e.g., [6], [2] and references therein for precise statements). We shall only use a very weak statement about Golod-Shafarevich groups:

Proposition 2.1. *If a group G is Golod-Shafarevich with respect to p , then its pro- p completion $G_{\widehat{p}}$ is not virtually abelian.*

Proposition 2.1 follows, for instance, from a theorem of Wilson [5] which asserts that every Golod-Shafarevich group has an infinite torsion quotient.

Proof of Theorem 1.2. Let G be a Golod-Shafarevich group, so that G has a presentation $\langle X | R \rangle$ with $1 - H_X(t) + H_R(t) < 0$ for some $t \in (0, 1)$, and let $\varepsilon = -(1 - H_X(t) + H_R(t))$.

Let $k_0 \in \mathbb{N}$ be such that $t^{k_0} < \frac{\varepsilon}{8}$. By Proposition 2.1 we can choose $x_1, y_1 \in \omega_{k_0} G$ whose images in $G_{\widehat{p}}$ do not commute. Then there exists $k_1 > k_0$ such that x_1 and y_1 do not commute modulo $\omega_{k_1} G$. By making k_1 larger we can also assume that $t^{k_1} < \frac{\varepsilon}{16}$.

Let S_1 be a finite generating set for $\omega_{k_1} G$, let $R_1 = \{[x_1, s], [y_1, s] : s \in S_1\}$ and $G_1 = G / \langle R_1 \rangle^G$. Note that if $\bar{x}_1$ and $\bar{y}_1$ are the images of x_1 and y_1 in G_1 , then

- (i) By property (B) above we have $G / \omega_{k_1} G \cong G_1 / (\omega_{k_1} G / \langle R_1 \rangle^G) = G_1 / \omega_{k_1} G_1$, so $\bar{x}_1$ and $\bar{y}_1$ do not commute modulo $\omega_{k_1} G_1$;
- (ii) $\bar{x}_1$ and $\bar{y}_1$ lie in the FC-radical of G_1 (and the same is true for any quotient of G_1).

The group G_1 need not be Golod-Shafarevich, but it surjects onto the group $\widehat{G}_1 = G_1 / \langle x_1, y_1 \rangle^{G_1}$ which is Golod-Shafarevich by construction.

Thus, the pro- p completion of G_1 is not virtually abelian, so we can find elements $x_2, y_2 \in \omega_{k_1} G$ and $k_2 > k_1$ such that the images of x_2 and y_2 in G_1 do not commute modulo $\omega_{k_2} G_1$ and $t^{k_2} < \frac{\varepsilon}{32}$.

Let S_2 be a finite generating set for $\omega_{k_2}G$, let $R_2 = \{[x_2, s], [y_2, s] : s \in S_2\}$ and $G_2 = G/\langle R_1 \cup R_2 \rangle^G$. By construction we have $G_1/\omega_{k_2}G_1 \cong G_2/\omega_{k_2}G_2$, the images of x_2 and y_2 in G_2 lie in the FC-radical of G_2 , and G_2 surjects onto a Golod-Shafarevich group.

Continuing this process indefinitely we obtain a sequence of groups $G = G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$, elements $\{x_i, y_i\}_{i \in \mathbb{N}}$ of G and integers $k_0 < k_1 < k_2 < \dots$ s.t.

- (i) G_{i+1} is a quotient of G_i for all i
- (ii) G_i surjects onto the group $G/\langle \cup_{j=1}^i \{x_j, y_j\} \rangle^G$
- (iii) x_i and y_i lie in $\omega_{k_{i-1}}G$, and $t^{k_{i-1}} < \frac{\varepsilon}{2^{i+2}}$
- (iv) The images of x_i and y_i in $G_{i-1}/\omega_{k_i}G_{i-1}$ do not commute
- (v) $G_{i-1}/\omega_{k_i}G_{i-1} \cong G_j/\omega_{k_i}G_j$ for all $j \geq i$
- (vi) The images of x_i and y_i in G_i lie in the FC-radical of G_i

Now let G_∞ be the inductive limit of $\{G_i\}$; in other words, if $G_i = G/N_i$, we let $N_\infty = \cup_{i \in \mathbb{N}} N_i$ and $G_\infty = G/N_\infty$. Let Q be the image of G_∞ in its pro- p completion, that is, $Q = G_\infty / \cap_{n \in \mathbb{N}} \omega_n G_\infty$.

Condition (ii) implies that G_∞ surjects onto the group $G/\langle \cup_{j=1}^\infty \{x_j, y_j\} \rangle^G$ which is Golod-Shafarevich by (iii). Thus, by Proposition 2.1 the group Q is infinite. Since Q is a subset of $(G_\infty)_{\widehat{p}}$, it is also residually finite.

Let $\pi : G \rightarrow Q$ be the natural projection. By condition (vi) the FC-radical of Q contains the subgroup H generated by the elements $\{\pi(x_i), \pi(y_i)\}_{i \in \mathbb{N}}$. It remains to show that H is not virtually abelian. Suppose not, so H contains a finite index abelian subgroup A . Then there exists integers $i < j$ such that $\pi(x_i x_j^{-1}) \in A$ and $\pi(y_i y_j^{-1}) \in A$. Conditions (iv) and (v) imply that $\pi(x_i)$ and $\pi(y_i)$ do not commute modulo $\omega_{k_i}Q$. Thus, if φ_i is the projection map $Q \rightarrow Q/\omega_{k_i}Q$, then $\varphi_i \pi([x_i, y_i]) \neq 1$. On the other hand, by construction $x_j, y_j \in \omega_{k_i}G$, so $\varphi_i \pi(x_j) = \varphi_i \pi(y_j) = 1$. Therefore,

$$\varphi_i \pi([x_i, y_i]) = \varphi_i \pi([x_i x_j^{-1}, y_i y_j^{-1}]) \in \varphi_i([A, A]) = \{1\}.$$

The obtained contradiction shows that the FC-radical of Q is not virtually abelian, which finishes the proof. $\square$

REFERENCES

- [1] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, *Analytic pro- p groups*. Second edition. Cambridge Studies in Advanced Mathematics, 61. Cambridge University Press, Cambridge, 1999.
- [2] M. Ershov, *Golod-Shafarevich groups with property (T) and Kac-Moody groups*, Duke Math. J. 145 (2008), no. 2, 309–339
- [3] S. A. Jennings, *The structure of the group ring of a p -group over a modular field*. Trans. Amer. Math. Soc. 50 (1941), 175–185.
- [4] S. Popa and S. Vaes, *On the fundamental group of II_1 factors and equivalence relations arising from group actions*, Quanta of maths, 519–541, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010.
- [5] J. Wilson, *Finite presentations of pro- p groups and discrete groups*. Invent. Math. 105 (1991), no. 1, 177–183.
- [6] E. Zelmanov, *Infinite algebras and pro- p groups*. Infinite groups: geometric, combinatorial and dynamical aspects, 403–413, Progr. Math., 248, Birkhäuser, Basel, 2005.

UNIVERSITY OF VIRGINIA, DEPARTMENT OF MATHEMATICS, P.O.BOX 400137, CHARLOTTESVILLE, VA 22904

E-mail address: ershov@virginia.edu