Math 8852. Unitary representations and property (T).
Problem Set 5.

1 (backlog from Homework #4). Let G be a locally compact abelian group, and fix $\chi \in \widehat{G} = Hom(G, S^1)$. Given $\epsilon > 0$ and a compact subset K of G, let $\mathcal{U}_{\chi,K,\epsilon}$ be the set of $\varphi \in \widehat{G}$ such that $|\varphi(k) - \chi(k)| < \epsilon$ for all $k \in K$. Prove that the sets $\mathcal{U}_{\chi,K,\epsilon}$ form a base for the Fell topology on $\widehat{G}$ (considered as the unitary dual of G). Deduce that the Fell topology on $\widehat{G}$ coincides with the open-compact topology on $\widehat{G}$ (considered as the Pontryagin dual of G).

2 (backlog from Homework #4). Let Σ_n be the set of elementary matrices in $SL_n(\mathbb{Z})$ of the form $E_{ij}(1)$. Use the proof of property (T) for $SL_n(\mathbb{Z})$, $n \geq 3$, given in class to find an explicit lower bound for the Kazhdan constant $\kappa(SL_n(\mathbb{Z}), \Sigma_n)$.

3. Let G be an amenable group. Prove that $\kappa(G, Q, L^2(G)) = 0$ for any compact subset Q of G (using the definition of amenability in terms of Fölner sets, given in class)

4. Let H be a closed subgroup of a locally compact group G such that G/H admits a finite regular invariant measure. In Lecture 22 we proved that G has property (T) $\iff$ H has property (T), but the proof of the forward implication used continuity of induction (Theorem 22.8) as a black box. Give a "direct" proof of this implication in the case when H is a finite index subgroup of G.

Hint: Let T be a left transversal for H in G, so that every element of G is uniquely written as th with $h \in H$ and $t \in T$. Now let (π, V) be a representation of H, and consider the induced representation $(\mathrm{ind}_H^G \pi, W)$ of G (where $W = L^2(G, H, V)$). Given $v \in V$, define an element $f_v \in W$ by $f_v(th) = \pi(h^{-1})v$. Then show that for any $\epsilon > 0$ and a compact subset K of G there exists $\delta > 0$ and a compact subset K' of H such that if v is (K', δ)-invariant, then f_v is (K, ϵ)-invariant. Deduce that if $1_H \prec \pi$, then $1_G \prec \mathrm{ind}_H^G$ (after which one can finish the proof as in class, but without using the black box).

5. Let $G = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and $N = SL_2(\mathbb{Z}) \lhd G$. Recall that we have a natural action of G on $\widehat{N}$ given by

$$(g * \chi)(n) = \chi(g^{-1}ng) \text{ for } g \in G, n \in N \text{ and } \chi \in \widehat{N}.$$

As in class, identify $\widehat{N}$ with the subset $\mathbb{T} := (-1/2, 1/2]^2$ of $\mathbb{R}^2$ where the point (x, y) corresponds to the character $(m, n) \mapsto e^{2\pi i(xm+yn)}$. Given a point $P \in \mathbb{R}^2$, let $\overline{P}$ be the unique element of $\mathbb{T}$ such that $P - \overline{P} \in \mathbb{Z}^2$. Prove that under the above identification, the action of $SL_2(\mathbb{Z})$ on $\mathbb{T}$ is given by

$$g * P = \overline{Ad(g) \cdot P},$$

where $Ad(g) = (g^{-1})^t$ and the dot on the right-hand side denotes the usual action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$ by left multiplication.