Math 8852. Unitary representations and property (T).
Problem Set 4.

1. backlog: Problem 2 from Homework #3 (realizing irreducible representations of the Heisenberg group as induced representations)

2. backlog: Problem 3 from Homework #2 (outline of the proof of the spectral measure decomposition for unitary representations of locally compact abelian groups). This will be used in the proof of relative property (T) of $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$.

3. Let G be a locally compact abelian group. Prove that the Fell topology on $\widehat{G}$ (considered as the unitary dual of G) concides with the open-compact topology on $\widehat{G}$ (considered as the Pontryagin dual of G).

4. Let $B_1, \ldots, B_k$ be a finite collection of subsets of a group G. Prove that if (G, B_i) has relative property (T) for each i, then $(G, \cup_{i=1}^k B_i)$ has relative property (T).

5. Let Σ_n be the set of elementary matrices in $SL_n(\mathbb{Z})$ of the form $E_{ij}(1)$. Use the proof of property (T) for $SL_n(\mathbb{Z})$, $n \geq 3$, given in class to find an explicit lower bound for the Kazhdan constant $\kappa(SL_n(\mathbb{Z}), \Sigma_n)$. You will need an explicit lower bound $\kappa(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2; \Sigma) \geq 1/10$ (to be proved later in class) for the relative Kazhdan constant $\kappa(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2; \Sigma)$, where Σ is the union of $\Sigma_2 \subset SL_2(\mathbb{Z})$ and standard basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of $\mathbb{Z}^2$ (where $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$ are canonically embedded in $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$).

6. Let H and K be subgroups of the same group, and assume that H normalizes K. Prove that $orth(H, K) = 0$.

7. A representation (π, V) of a topological group G is called *cyclic* if there exists $v \in V$ such that there are no proper closed $\pi(G)$-invariant subspaces of V containing v. Prove that every unitary representation is a direct sum of cyclic subrepresentations. **Hint:** Consider the set of all pairwise orthogonal collections of cyclic subrepresentations of V, ordered by inclusion, and apply Zorn's lemma.

8. Let $\mathbb{F}_p$ be a finite field of prime order p and $G = Heis(\mathbb{F}_p)$ thought of as the group of upper-unitriangular matrices in $GL_3(\mathbb{F}_p)$. Let $x = E_{12}(1)$ and $y = E_{23}(1)$, and let $H = \langle x \rangle$ and $K = \langle y \rangle$. Prove that $orth(H, K) = \frac{1}{\sqrt{p}}$.