Math 8852. Unitary representations and property (T).
Problem Set 3.

Below [BHV] refers to the book 'Kazhdan's property (T)' by Bekka, de la Harpe and Valette.

1. (leftover from Problem Set 2). Let $G = H(\mathbb{Z}) = \langle x, y, z \mid z = [x, y], [x, z] = [y, z] = 1\rangle$ be the Heisenberg group over $\mathbb{Z}$. Let (π, V) be an infinite-dimensional **irreducible** unitary representation of $H(\mathbb{Z})$, which has an eigenvector for y. Prove that there exists an orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$ of V and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| = |\mu| = 1$ and λ not a root of unity such that $\pi(x)(e_i) = e_{i+1}$ and $\pi(y)(e_i) = \mu\lambda^i e_i$ for all i.

Comments and hint: At the last problem session we showed that there exist unit vectors $\{e_i\}_{i\in\mathbb{Z}}$ which topologically span V and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| = |\mu| = 1$ such that x and y act in the desired way. Assuming that λ is not a root of unity, we also deduced that $\{e_i\}$ are pairwise orthogonal, thus finishing the proof. It remains to eliminate the possibility that λ is a root of unity. Here is a hint for how to do this.
Suppose that λ is a root of unity of order n. For each $0 \leq i \leq n-1$ let V_i be the closure of the span of $\{e_{i+nk} : k \in \mathbb{Z}\}$. First note that both y and z act as scalars on each V_i and show that $V = \oplus_{i=0}^{n-1} V_i$. Also note that each V_i is x^n-invariant. Next explain why there exists a proper nonzero subspace $W \subseteq V_0$ which is x^n-invariant, and then use W to construct a non-trivial (closed) G-invariant subspace of V.

2. Let $G = H(\mathbb{Z})$ be as in Problem 1 and H the subgroup of G generated by y and z. Prove that each of the representations of G described in Problem 1 is equivalent to the induced representation $\mathrm{Ind}_H^G \sigma$ for some one-dimensional representation σ of H. **Hint:** One-dimensional representations of H are all of the form $\sigma_{\lambda,\mu}$ with $|\lambda| = |\mu| = 1$ where $\sigma_{\lambda,\mu}(z^i y^j)$ is the multiplication by $\lambda^i \mu^j$.

3. In class we outlined the proof of the "Induction by stages" theorem which asserts that given closed subgroups $K \subseteq H \subseteq G$ of a topological group G and a unitary representation (π, V) of G we have

$$\mathrm{Ind}_K^G \pi \cong \mathrm{Ind}_H^G (\mathrm{Ind}_K^H \pi) \text{ as representations of } G.$$

The goal of this exercise is to fill in the details that we omitted in class, thus providing a complete proof of the theorem, in the case when G is discrete (in general, there are several non-trivial analytical aspects that need to be taken care of).

Below we recall the notations introduced in class, and from now on we assume that G is discrete (thus integrals are replaced by the sums).

Let $V_K^G = L^2(G/K, V)$ be the representation space of $\operatorname{Ind}_K^G \pi$, that is, V_K^G is the space of functions $f : G \to V$ satisfying the K-equivariance condition

$$f(gk) = \pi(k^{-1})f(g) \text{ for all } k \in K \text{ and all } g \in G$$

and such that $\|f\|_{V_K^G} < \infty$, where by definition

$$\|f\|_{V_K^G}^2 = \sum_{gK \in G/K} \|f(g)\|^2.$$

Also recall that the inner product on V_K^G is given by

$$\langle f, f' \rangle = \sum_{gK \in G/K} \langle f(g), f'(g) \rangle_V.$$

Similarly, let $V_K^H = L^2(H/K, V)$ be the representation space of $\operatorname{Ind}_K^H \pi$, let $W_K^G = L^2(G/H, V_K^H) = L^2(G/H, L^2(H/K, V))$ be the representation space of $\operatorname{Ind}_H^G(\operatorname{Ind}_K^H \pi)$, and let $\rho : G \to \mathcal{U}(W_K^G)$ denotes the action of G on W_K^G. In class we defined the map $\Phi : V_K^G \to W_K^G$ by

$$((\Phi(f))(g))(h) = f(gh) \text{ for all } f \in V_K^G, g \in G, h \in H,$$

and we claimed that Φ is a unitary equivalence between the representations Ind_K^G and $\operatorname{Ind}_H^G(\operatorname{Ind}_K^H \pi)$. The parts that we did not explicitly verified in class were

(a) Φ intertwines the above representations, that is, for all $g \in G$

$$\Phi \circ \operatorname{Ind}_K^G \pi(g) = \rho(g) \circ \Phi \text{ as maps from } V_K^G \text{ to } W_K^G.$$

(b) Φ is bijective.

Hint for (b): Let S be a left transversal (that is, a set of left coset representatives) for K in H, and let T be a left transversal for H in G. Then $TS = \{ts : t \in T, s \in S\}$ is a left transversal for K in G, and every element of TS is uniquely represented as ts with $t \in T$ and $s \in S$. Prove that there are natural identifications (which are actually isometries of Hilbert spaces) between V_K^G and $L^2(TS, V)$ (here L^2 is in the usual sense) and between W_K^G and

$L^2(T, L^2(S, V))$, and that under these identifications the map Φ corresponds to the map $\Psi : L^2(TS, V) \to L^2(T, L^2(S, V))$ given by

$$((\Psi(f))(t))(s) = f(ts).$$

Thus, we are reduced to proving that Ψ is bijective, which is easy. In fact, in the course of the proof you will automatically show that Ψ (and hence also Φ) is an isometry, something that we established in class without assuming that G is discrete (but modulo some hand-waving).

4. Let (π, V) be a unitary representation of a group G and S a subset of G.

(a) Let $S^{-1} = \{s^{-1} : s \in S\}$. Prove that

$$\kappa(G, S, \pi) = \kappa(G, S^{-1}, \pi) = \kappa(G, S \cup S^{-1}, \pi).$$

(b) Let $n \in \mathbb{N}$, and let S^n be the set of elements of G which are representable as $s_1 \ldots s_m$ with $m \leq n$ and $s_i \in S \cup S^{-1}$. Prove that

$$\kappa(G, S^n, \pi) \leq n \cdot \kappa(G, S, \pi).$$

5. Let Q be a compact subset of a topological group G, and suppose that there exists $\epsilon > 0$ such that $\kappa(G, Q, \pi) \geq \epsilon$ for every non-trivial irreducible representation π of G.

(a) Suppose that G is discrete (and hence Q is finite). Prove that $\kappa(G, Q, \pi) \geq \frac{\epsilon}{\sqrt{|Q|}}$ for every completely reducible representation π of G without nonzero invariant vectors. **Hint:** Consider the quantity $K(G, Q, \pi) = \inf_{v \in V, v \neq 0}(\sum_{q \in Q} \frac{\|\pi(q)v - v\|^2}{\|v\|^2})$ and note that $\kappa(G, Q, \pi)^2 \leq K(G, Q, \pi) \leq |Q| \cdot \kappa(G, Q, \pi)^2$.

(b) In the general case prove that there exists $\delta > 0$ such that $\kappa(G, Q, \pi) \geq \delta$ for every completely reducible representation π of G without nonzero invariant vectors. Start by showing that if there is no such δ, then there actually exists a representation π in the above class with $\kappa(G, Q, \pi) = 0$ and then use an idea similar to part (a) to reach a contradiction.

Remark: If G is locally compact the assertions of (a) and (b) remain true for an arbitrary (not necessarily completely reducible) representation π without nonzero invariant vectors, but the proof requires more advanced tools. If in addition G is second countable and the representation space of π is separable, these generalizations of (a) and (b) can be proved by essentially the same method using decomposition of π into a direct integral of irreducibles (see [BHV, F.6]).

6. In class we proved that $\kappa(G, G) \geq \sqrt{2}$ for any group G. This problem investigates how far this inequality is from being optimal.

(a) Prove that $\min_{[\pi] \in \widehat{\mathbb{Z}} \setminus \{1_{\mathbb{Z}}\}} \kappa(\mathbb{Z}, \mathbb{Z}, \pi) = \sqrt{3}$ (where as usual $\widehat{G}$ is the unitary dual of G) and that the minimum is attained on $[\pi] = e^{2\pi i/3}$ (where we use natural identification of $\widehat{\mathbb{Z}}$ and S^1).

(b) Prove that $\kappa(G, G) = \sqrt{2}$ for any non-compact locally compact group G. **Hint:** In view of the general inequality $\kappa(G, G) \geq \sqrt{2}$, it suffices to find a representation π of G without nonzero invariant vectors such that $\kappa(G, G, \pi) = \sqrt{2}$.

(c) Prove that if G is a finite group of order n, then $\kappa(G, G) \geq \sqrt{\frac{2n}{n-1}}$

(d) Prove that inequality in (c) is actually equality for $G = \mathbb{Z}/2\mathbb{Z}$ and $G = \mathbb{Z}/3\mathbb{Z}$.