Math 8852. Unitary representations and property (T).

Problem Set 2.

Below [BHV] refers to the book 'Kazhdan's property (T)' by Bekka, de la Harpe and Valette.

1. Let $H(\mathbb{Z}) = \langle x, y, z \mid z = [x, y], [x, z] = [y, z] = 1\rangle$ be the Heisenberg group over $\mathbb{Z}$. Let (π, V) be an infinite-dimensional unitary representation of $H(\mathbb{Z})$, which has an eigenvector for y. Prove that there exists an orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$ of V and $\lambda, \mu\mathbb{C}$ with $|\lambda| = |\mu| = 1$ and λ not a root of unity such that $\pi(x)(e_i) = e_{i+1}$ and $\pi(y)(e_i) = \mu\lambda^i e_i$ for all i.

2. Let X be a locally compact space, V a Hilbert space and E an (X, V)-spectral measure. Prove that the map $f \mapsto \int\limits_X f(x)dE(x)$ from $L^\infty(X)$ to $\mathcal{L}(V)$ is a homomorphism of $*$-algebras with 1.

3. Let G be a locally compact abelian group and (π, V) an irreducible unitary representation of G. Use the following outline to prove that there exists a regular $(\widehat{G}, V)$-spectral measure E such that

$$\pi(g) = \int\limits_{\widehat{G}} \chi(g)dE(\chi) \text{ for all } g \in G. \qquad (***)$$

(a) Verify that for any $v \in V$, the matrix coefficient function $c^\pi_{v,v} : G \to \mathbb{C}$ (given by $c^\pi_{v,v}(g) = \langle\pi(g)v, v\rangle$) is positive-definite (see page 373 for the definition). Hence by Bochner's theorem there exists a finite positive regular measure μ_v on $\widehat{G}$ such that $c^\pi_{v,v}(g) = \int\limits_{\widehat{G}} \chi(g)d\mu_v(\chi)$

(b) Given $u, v \in V$, define a complex measure $\mu_{u,v}$ on $\widehat{G}$ by

$$\mu_{u,v} = \frac{1}{4}(\mu_{u+v} - \mu_{u-v} + i\mu_{u+iv} - i\mu_{u-iv}).$$

Prove that $c^\pi_{u,v}(g) = \int\limits_{\widehat{G}} \chi(g)d\mu_{u,v}(\chi)$

(c) Using Riesz Representation Theorem, show that for every $B \in \mathcal{B}(X)$, there exists a (well-defined) bounded operator $E(B) \in \mathcal{L}(V)$ such that $\langle E(B)u, v\rangle = \mu_{u,v}(B)$ for all u, v. Prove that E is a regular $(\widehat{G}, V)$-spectral measure.

(d) Now verify that the spectral measure E constructed in (c) satisfies (***).

(e) In class we checked that for any $(\widehat{G}, V)$-spectral measure E, the formula (***) defines a possibly non-continuous unitary representation of G, but we did not show that regularity of E implies that π is strongly continuous. Prove this fact (or read the proof on page 399 of [BHV]).

4. Let G be a locally compact abelian group. Prove that if G is compact, then $\widehat{G}$ is discrete, and vice versa, if G is discrete, then $\widehat{G}$ is compact.

5. Let G be a locally compact group, μ a left Haar measure on G and Δ the modular function of G. Prove that

(a) $\int\limits_G f(xg)d\mu(x) = \Delta(g^{-1}) \int\limits_G f(x)d\mu(x)$ for any $g \in G$ and $f \in C_c(G)$

(b) $\int\limits_G f(x^{-1})d\mu(x) = \int\limits_G \Delta(x^{-1})f(x)d\mu(x)$ for any $f \in C_c(G)$. (If you do not succeed, see Lemma A.3.4 in [BHV]).

6. Prove that discrete groups are unimodular.

7. Let G be a locally compact group and H a closed subgroup of G, and suppose that H is unimodular.

(a) Assume that H is contained in $\overline{[G,G]}$, the closure of the commutator subgroup of G. Prove that G/H admits an invariant measure.

(b) Now assume that G is not unimodular and moreover H is not contained in $\operatorname{Ker} \Delta_G$ (where Δ_G is the modular function of G). Prove that G/H admits a semi-invariant measure, but does not admit an invariant measure. Give a specific example where the above hypotheses are satisfied.