Math 8852. Unitary representations and property (T).
Problem Set 1.

Below [BHV] refers to the book 'Kazhdan's property (T)' by Bekka, de la Harpe and Valette.

1. Let G be a locally compact group.

   (a) Prove that the left-regular representation $(\lambda, L^2(G))$ is strongly continuous, that is, the map $g \mapsto \lambda(g)f$ is continuous for any $f \in L^2(G)$.

   (b) Prove that the map $g \mapsto \lambda(g)$ is not necessarily continuous in the uniform operator topology.

2. Verify basic properties of matrix coefficients which were used in the proof of Peter-Weyl theorem:

   (i) $c^{\pi_1}_{u_1,v_1} + c^{\pi_2}_{u_2,v_2} = c^{\pi_1\oplus\pi_2}_{u_1\oplus u_2, v_1\oplus v_2}$

   (ii) $c^{\pi_1}_{u_1,v_1} \cdot c^{\pi_2}_{u_2,v_2} = c^{\pi_1\otimes\pi_2}_{u_1\otimes u_2, v_1\otimes v_2}$

   (iii) $\overline{c^{\pi}_{u,v}} = c^{\overline{\pi}}_{u,v}$

3. Let (π, V) be a unitary representation of a group G. Prove that the conjugation representation $(\overline{\pi}, \overline{V})$ is equivalent to the contragradient representation (π^*, V^*) defined as follows: $V^* = \mathcal{L}(V, \mathbb{C})$ is the dual space of V and $(\pi^*(g)f)(v) = f(\pi(g^{-1})v)$ for all $v \in V$, $f \in V^*$ and $g \in G$.

4. Prove that two unitary representations are equivalent if and only if they are unitarily equivalent. **Note:** A more general result is proved on page 307 of [BHV]. The goal of this exercise is to write down a simplified version of that argument sufficient to solve this problem.

5. Let $H(\mathbb{Z}) = \langle x, y, z \mid z = [x, y], [x, z] = [y, z] = 1\rangle$ be the Heisenberg group over $\mathbb{Z}$. Let V be a Hilbert space with orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$. As in class, given $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, let π_λ be the (unitary) representation of G on V given by $\pi_\lambda(x)e_i = \lambda^i e_i$ and $\pi_\lambda(y)e_i = e_{i+1}$. In class we proved that π_λ is irreducible if λ is not a root of unity. Prove that if λ is a root of unity, then π_λ is not irreducible. Can you explicitly construct a G-invariant subspace?

6. Let (π, V) be a representation of a group G.

(a) Given a subgroup H of G, let $V^H = \{v \in V : \pi(h)v = v \text{ for all } h \in H\}$ be the subspace of H-invariant vectors. Prove that if H is normal in G, then V^H is G-invariant.

(b) Assume that π is unitary and irreducible. Prove that every element of $Z(G)$ (the center of G) acts as scalar multiplication.

7. Let G be a compact group, normalize the Haar measure on G so that $\mu(G) = 1$, and let (π, V) be a unitary representation of G.

(a) Prove that there exists a (uniquely defined) operator $\int\limits_G \pi(g)dg \in \mathcal{L}(V)$ by

$$\langle (\int\limits_G \pi(g)dg)v, w\rangle = \int\limits_G \langle \pi(g)v, w\rangle dg$$

for all $v, w \in V$.

(b) Let $W = V^G$ be the subspace of G-invariant vectors. Prove that $\int\limits_G \pi(g)dg$ is the orthogonal projection onto W.

8. Exercises A.8.5 and A.8.9 in [BHV] (pp. 340-341).