Math 8851. Homework #5. To be completed by 6pm on Thu, Apr 10

1. Problem 6 from HW#4.

2. Problem 7 from HW#4.

Recall that in Lecture 23 we proved that for any hyperbolic group G, any torsion subgroup T of G is finite and moreover is conjugate to a subgroup of a ball of a fixed radius (in fact, the radius only depends on the hyperbolicity constant of G). That proof was fairly long and quite technical. If we only wanted to prove that any finite subgroup of G is conjugate to a subgroup of a ball of a fixed radius, that could be done with much less work. The next two problems outline a simpler proof of the latter fact.

We start with the definition of a center of a bounded subset. Let X be a proper metric space and A a bounded subset of X. For any $x \in X$ let $r_A(x)$ be the minimum $r \in \mathbb{R}$ such that A is contained in $B_r(x)$, the ball of radius r centered at x (it is easy to see that such minimum always exists). Then $r_A(x)$ considered as a function of x is continuous and goes to ∞ if $x \to \infty$, so by properness of X, it attains a minimum which we denote by r_A and call the *radius of* A. A *center of* A is any point such that $r_A(x) = r_A$. The set of centers of A is denoted by $Cent(A)$ (note that a set can have more than one center).

3. Let X be a proper hyperbolic geodedic space satisfying $Hyp_{slim}(\delta)$, and let A be a bounded subset of X. Prove that for any $x, y \in Cent(A)$ we have $d(x, y) \leq 4\delta$.

Hint: choose a geodesic $[x, y]$, and let m be its midpoint. By definition of r_A, there exists $a \in A$ such that $d(a, m) \geq r_A$. Consider a geodesic triangle $[x, y, a]$ and apply $Hyp_{slim}(\delta)$ condition to the point $m \in [x, y]$. After some calculations you should be able to prove that either $d(x, m) \leq 2\delta$ or $d(y, m) \leq 2\delta$ (depending on whether m is closer to $[x, a]$ or $[y, a]$). Since m is the midpoint of $[x, y]$, either inequality implies that $d(x, y) \leq 4\delta$.

4. Now let G be a hyperbolic group, S some finite generating set of G, and suppose that $X = Cay(G, S)$ satisfies $Hyp_{slim}(\delta)$. Prove that F is conjugate to a subgroup of $B_{4\delta+1}(e)$.

Hint: Let x be any center of F in X (it need not be a vertex) and choose $g \in G$ such that $d(g, x) \leq \frac{1}{2}$. Let $A = g^{-1}F$ and $H = g^{-1}Fg$. Prove that

(i) A has a center y with $d(y,e) \leq \frac{1}{2}$
(ii) the set $Cent(A)$ is invariant under left-multiplication by any $h \in H$.

Then deduce from (i),(ii) and Problem 3 that $H \subseteq B_{4\delta+1}(e)$.

The next problem deals with the topology on the boundary of a hyperbolic space X. We start by recalling some notations from class. Fix a base point $p \in X$. Let $Geo_p(X)$ denote the set consisting of all geodesic rays $\gamma : [0,\infty) \to X$ and all geodesic paths $\gamma : [0,T] \to X$ with $\gamma(0) = p$, all parameterized with respect to arc length. Extend each geodesic path $\gamma : [0,T] \to X$ to a map defined on $[0,\infty)$ by setting $\gamma(t) = \gamma(T)$ for all $t > T$.

Next fix $K > 2\delta$. Given $\gamma \in Geo_p(X)$ and $n \in \mathbb{N}$, define $V_n(\gamma)$ to be the set of all $\alpha \in Geo_p(X)$ such that $d(\gamma(n), \alpha(n)) < K$. Lemma 25.5 from class (which we did not prove) asserts that for a fixed γ, the images of the sets $V_n(\gamma)$, $n \in \mathbb{N}$, in ∂X, form a base of (not necessarily open) neighborhoods of $\gamma(\infty)$ in ∂X.

Note that $V_n(\gamma)$ is clearly open in $Geo_p(X)$, so if the relation $\sim$ on $Geo_p(X)$ is trivial (which is the case in both examples in Problem 5 below), the image of $V_n(\gamma)$ in ∂X is also open.

5.

(a) Let $X = \mathbb{H}^2$ in the upper half-plane model, $p = (1,0)$ and γ the geodesic ray starting at p and going straight down (the boundary point represented by γ is $(0,0)$). Set $K = 2$ (this satisfies $K > 2\delta$, but in fact when there are no equivalent geodesics, any $K > 0$ could be used). For each $n \in \mathbb{N}$ compute the set $V_n(\gamma) \cap \partial\mathbb{H}^2$ – drawing the picture will likely be helpful. You may use without proof that any circle in $\mathbb{H}^2$ is also a Euclidean circle (albeit with a different center).

(b) Now let $X = T_d$, the regular tree of degree $d \geq 3$. Fix a vertex p of X, and as in class, view X as a tree rooted at p, drawn upside down, withe the root at the top. Thus, elements of $Geo_p(X)$ are precisely downwards paths (finite or infinite). Since $\delta = 0$ in this example, we can assume that $K < 1$. Describe explicitly the sets $V_n(\gamma)$ for a geodesic ray γ.

(c) Now use your answer in (b) and Lemma 25.5 from class to prove that ∂T_d is homeomorphic to a countable product of finite sets of cardinality ≥ 2 (it is known that any such product is homeomorphic to the Cantor set).

6. Let G be a hyperbolic group and $g \in G$ an element of infinite order. Prove that

(a) the elementary subgroup $E(g)$ is self-normalizing in G, that is, if $hE(g)h^{-1} = E(g)$ for some $h \in G$, then $h \in E(g)$. **Hint:** Use the fact that $\langle g \rangle$ has finite index in $E(g)$.

(b) Prove that $hE(g)h^{-1} = E(hgh^{-1})$ for any $h \in G$.

As an immediate consequence of (b), we deduce that for any infinite hyperbolic group G, either G is equal to $E(g)$ for some g (and hence G is virtually cyclic) or G contains infinite order elements g and k such that $E(g) \neq E(k)$.

7. Use boundaries to show that a free group or any rank cannot be quasi-isometric to any surface group

$$S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$