Math 8851. Homework #3. To be completed by 6pm on Thu, Mar 6

1. (Problem 5 from HW#2) Let $G = \mathbb{Z}^2$, consider its standard presentation $G = \langle a, b \mid [a, b] = 1\rangle$, and let δ be the associated function.

(a) Prove that $\delta(n) \leq \binom{n}{2}$ for all n.
(b) Find a specific constant $K > 0$ such that $\delta(n) \geq Kn^2$ for all sufficiently large n.
(c) How do the answers to (a) and (b) change if we consider $\mathbb{Z}^n$ for some $n > 2$ with the presentation $\langle a_1, \ldots, a_n \mid [a_i, a_j] = 1 \text{ for all } i < j\rangle$?

Hint: For (a) first describe a simple algorithm which reduces any $w \in F(\{a, b\})$ such that $w =_G 1$ to the identity element in at most $\binom{n}{2}$ steps where $n = \|w\|$. It should be somewhat similar to Dehn's algorithm. Then deduce that $Area(w) \leq \binom{n}{2}$ The argument for this part should be similar to your solution to HW#2.4. One way to solve (b) is to follow the proof of "(4)⇒ (1)" in Theorem 10.4 (since you are dealing with a specific and very simple presentation, you can give better bounds than the proof in the general case).

2. (Problem 6 from HW#2 with an extended hint). Let G be a hyperbolic group. Prove that G has only finitely many conjugacy classes of torsion elements (that is, elements of finite order).

Hint: Fix a Dehn presentation (X, R) for G (which exists by Theorem 10.4). Let C be a conjugacy class of torsion elements in G, let $g \in C$ be an element of smallest possible word length (with respect to X) and choose any word $w \in F(X)$ which represents g and such that $\|w\| = \|g\|$. Note that w must be cyclically reduced (explain why). Let n be the order of g, so that w^n is a relator of G. Apply a single step of Dehn's algorithm to w^n and deduce that one of the following must hold:

(i) some conjugate of g is representable by a word shorter than w.
(ii) some cyclic shift of w is a subword or some relator $r \in R^*$.

Note that case (i) is ultimately impossible as it would contradict the choice of g, so (ii) must always occur. Deduce that there are only finitely many choices for w and hence also for g and for C.

3. Let X be any finite set, let $w \in F(X)$ be a non-trivial word, and let $R = \{w^n\}$ for some $n \in \mathbb{N}$ (thus R is a 1-element set). Prove that the presentation (X,R) satisfies $C'(\frac{1}{n})$. **Note:** Recall that when we are searching for pieces of a presentation, we need to consider all relators from R^*, the symmetrization of R, not just R itself.

4. Prove Lemma 15.4 from class. Let $X = \{a, b\}$ be a 2-element set. Prove that for any real number $\alpha > 0$ and any $k, M \in \mathbb{N}$ there exist cyclically reduced words $u_1, \ldots, u_k \in F(X)$ such that $\|u_i\| > M$ for all i and $R = \{u_1, \ldots, u_k\}$ considered as a set of relations satisfies $C'(\alpha)$.

5. Let G be a non-cyclic group such that any generating set of G contains an element of order 2 (for example, G could be any dihedral group). Use Greendlinger's Lemma (either algebraic or geometric version) to prove that G does not admit any presentation satisfying $C'(\lambda)$ for $0 \le \lambda \le \frac{1}{6}$.

Before Problem 6, we introduce the definition of a Schreier transversal and state Schreier's theorem. Let F be a free group and fix a free generating set X for F.

Definition. Let H be a subgroup of F. A subset T of F is a called a (right) *Schreier transvseral* for H in F if

(a) T is a right transversal, that is, T contains exactly one element from each right coset Hg with $g \in G$;
(b) for any $t \in T$, any prefix of t (considered a word in $X \sqcup X^{-1}$) must also be in T. This includes the empty prefix, so in particular we must have $1 \in T$.

It is not difficult to show that every subgroup has a Schreier transversal. Below once a Schreier transversal T has been fixed, we will use the following notation: given $g \in F$, we will denote by $\overline{g}$ the unique element of T such that $Hg = H\overline{g}$.

Theorem (Schreier)**.** *Let F, X and H be as above, and let T be a Schreier transversal for H in F. For each $t \in T$ and $x \in X$ let $u_{t,x} = tx(\overline{tx})^{-1}$. Then the elements of the form $u_{t,x}$ which are not equal to 1 are all distinct and form a free generating set for H.*

6. Let $X = \{a, b\}$ be a 2-element set. Let $F = F(X)$, let $\pi : F \to \mathbb{Z}$ be the unique homomorphism such that $\pi(a) = 1$ and $\pi(b) = 0$, and let $H = \mathrm{Ker}(\pi)$.

(a) Prove that $T = \{a^i : i \in \mathbb{Z}\}$ is a Schreier transversal for H in F;
(b) Now use Schreier's theorem to find an explicit free generating set for H.