Math 8851. Homework #2. To be completed by 6pm on Thu, Feb 20

1 (extended version of HW#1.1). Let G be a group and S a generating set of G.

(a) Prove that the following are equivalent:
  (i) G is free and S is a free generating set of G. By definition this means that every element of G can be uniquely written as a reduced word $\prod_{i=1}^{n} s_i^{\varepsilon_i}$ with $s_i \in S$ and $\varepsilon_i = \pm 1$ (reduced means that $s_i \neq s_{i+1}$ whenever $\varepsilon_{i+1} = -\varepsilon_i$).
  (ii) The Cayley graph $Cay(G, S)$ is a tree and S has no elements of order 2.

(b) Describe all groups G with the property that $Cay(G, S)$ is a tree for some generating set S of G.

2. Let (X, R) be a group presentation, $G = \langle X|R\rangle$, and let $\mathcal{D}$ be a van Kampen diagram over (X, R). Prove that one can label the vertices of $\mathcal{D}$ by elements of G such that whenever e is an oriented edge from a vertex v to a vertex w we have $L(w) = L(v)L(e)$ (where $L(\cdot)$ denotes the label of a vertex or an edge). Moreover, show that if we fix a base vertex v_0, then $L(v_0)$ can be chosen to be any element of G, and once $L(v_0)$ is chosen, all other vertex labels are uniquely determined. **Hint:** Use van Kampen's lemma.

3. Let $X = \{a, b\}$, $R = \{aba^{-1}b^{-1}\}$ and $G = \langle X|R\rangle \cong \mathbb{Z}^2$.

(a) Let $w = a^2b^2a^{-2}b^{-2}$, and let $\mathcal{D}$ be the disk van Kampen diagram of area 4 from the example in Lecture 9 with $L(\partial\mathcal{D}) = w$. Use the proof of van Kampen's lemma to explicitly write w in the form $\prod_{i=1}^{4} u_i r_i^{\pm 1} u_i^{-1}$ with $u_i \in F(X)$ and $r_i = aba^{-1}b^{-1}$ (as the only element of R in this case).

(b) Now reverse the process from (a): start with the factorization found in (a), construct the corresponding 'lollipop' diagram, call it $\mathcal{D}'$, and show that after edge cancellations in $\partial\mathcal{D}'$ (as defined below), one obtains the original diagram $\mathcal{D}$ from (a).

Here is what we formally mean by an edge cancellation. Suppose that e_1 and e_2 are consecutive edges of $\partial\mathcal{D}'$ (as we traverse $\partial\mathcal{D}'$ in some direction) which have the same label $x \in X$ and point in opposite directions. As we traverse e_1, we move from some vertex u to some vertex v, and then as we traverse e_2, we move from v to some vertex w (which may coincide with u).

(i) If $w \neq u$, we start by gluing the edges e_1 and e_2, identifying u and w. If after this process the vertex $u = w$ becomes a leaf, we also remove the entire edge $e_1 = e_2$.

(ii) If $w = u$, we remove the edges e_1 and e_2 possibly together with any cells of $\mathcal{D}'$ enclosed between e_1 and e_2.

You should convince yourself that each of the operations (i) and (ii) results in a valid van Kampen diagram whose boundary label is obtained from $L(\partial\mathcal{D}')$ by the cancellation of the subword xx^{-1} or $x^{-1}x$ corresponding to the edges e_1 and e_2.

4. Prove that if (X, R) is a (finite) Dehn presentation of some group G and δ is the associated Dehn function, then $\delta(n) \leq n$ for all $n \in \mathbb{N}$.

5. Let $G = \mathbb{Z}^2$, consider its standard presentation $G = \langle a, b \mid [a, b] = 1\rangle$, and let δ be the associated function.

(a) Prove that $\delta(n) \leq \binom{n}{2}$ for all n.

(b) Find a specific constant $K > 0$ such that $\delta(n) \geq Kn^2$ for all sufficiently large n.

(c) How do the answers to (a) and (b) change if we consider $\mathbb{Z}^n$ for some $n > 2$ with the presentation $\langle a_1, \ldots, a_n \mid [a_i, a_j] = 1 \text{ for all } i < j\rangle$?

Hint: For (a) first describe a simple algorithm which reduces any $w \in F(\{a, b\})$ such that $w =_G 1$ to the identity element in at most $\binom{n}{2}$ steps where $n = \|w\|$. Then deduce that $Area(w) \leq \binom{n}{2}$ (the argument for this part should be similar to your solution to Problem 4). One way to solve (b) is to follow the proof of "(4)$\Rightarrow$ (1)" in Theorem 10.4 (since you are dealing with a specific and very simple presentation, you can give better bounds than the proof in the general case).

6. Let G be a hyperbolic group. Prove that G has only finitely many conjugacy classes of torsion elements (that is, elements of finite order). **Hint:** Let (X, R) be a Dehn presentation for G (which exists by Theorem 10.4). Prove that there are only finitely many torsion elements $g \in G$ for which there exists a cyclically reduced word w such that w represents g and $\|w\| = \|g\|$. Then argue that every conjugacy class of torsion elements must contain g with this property. To start you probably need to figure out why having a cyclically reduced representative is helpful.