Math 8851. Homework #9. To be completed by Thu, Apr 13

1. Fix a prime p. Recall that given a group presentation (X,R) where X is finite and R is possibly infinite we define the p-deficiency $def_p(X,R)\in\mathbb{R}\cup\{-\infty\}$ by

$$def_p(X,R)=|X|-1-\sum_{r\in R}\frac{1}{p^{\deg_{F(X)}(r)}}.$$

Recall that $\deg_{F(X)}(r)$ is the largest $k\in\mathbb{Z}_{\geq 0}$ such that $r=u^{p^k}$ for some $u\in F(X)$.

If G is a finitely generated group, we define $def_p(G)=\sup\{def_p(X,R)\}$ where (X,R) ranges over all presentations of G by generators and relators with X finite.

Now let G^p be the subgroup generated by all p^{th} powers $\{g^p : g\in G\}$ and consider (again assuming G is finitely generated) the quotient $G/[G,G]G^p$ where – it is a finite abelian group of exponent p and thus can be considered as a vector space over $\mathbb{F}_p$. Denote by $d_p(G)$ the dimension of this space (equivalently, $d_p(G)=\log_p[G:[G,G]G^p]$).

(a) Prove that $d_p(G)\geq def_p(G)+1$.
(b) Deduce that if $def_p(G)>-1$, then G has a normal subgroup of index p.
(c) Assume that H is a normal subgroup of G of p-power index. Prove that $def_p(H)\geq[G:H]def_p(G)$ (use Proposition 19.3 from class which asserts that this is true when H is normal of index p).
(d) Now assume that $def_p(G)>0$ and G is finitely presented. Use (a), (c) and Theorem 1.12 in the following paper of M. Lackenby

`http://arxiv.org/abs/math/0702571`

to prove that G has a finite index subgroup which homomorphically maps onto a non-abelian free group.

2. Let G be a group and $\{G_n\}_{n=1}^{\infty}$ a central series of G, that is, a descending chain of normal subgroup of G where $G_1=G$ and $[G_i,G_j]\subseteq G_{i+j}$ for all i and j. Recall the definition of the associate Lie ring $L(G)$.

As a set $L(G)=\bigoplus_{n=1}^{\infty}L_n(G)$ where $L_n(G)=G_n/G_{n+1}$. Elements of L which lie in $L_n(G)$ for some n are called homogeneous.

The addition on each $L_n(G)$ is simply the quotient group operation (note that G_n/G_{n+1} is abelian since $[G_n, G_n] \subseteq G_{2n} \subseteq G_{n+1}$).

The Lie bracket is defined as follows. First given homogeneous elements $u \in L_n(G)$ and $v \in L_m(G)$ we choose $g \in G_n$ and $h \in G_m$ such that $u = gG_{n+1}$ and $v = hG_{m+1}$ and set $[u, v] = [g, h]G_{n+m+1}$ where $[g, h] = g^{-1}h^{-1}gh$ is the group commutator of g and h.

Given arbitrary elements $u, v \in L$, we write them as sums of homogeneous components $u = \sum u_i$ and $v = \sum v_j$ and set $[u, v] = \sum\limits_{i,j}[u_i, v_j]$.

(a) Prove that the Lie bracket is well defined, that is, in the definition of $[u, v]$ in the homogeneous case the value is independent of the choice of g and h.

(b) Prove that $L(G)$ with the above operations is a Lie ring, that is, satisfies the following axioms:
   (1) $(L(G), +)$ is an abelian group.
   (2) $[x, y + z] = [x, y + z]$ and $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in L(G)$.
   (3) $[x, x] = 0$ for all $x \in L(G)$.
   (4) $[[x, y], z] + [[y, z], x] + [[z, y], x] = 0$ for all $x, y, z \in L(G)$.

   Hint: To prove (4) use the following group-theoretic identity called the Hall-Witt identity: $[[a, c], c^a][[c, a], b^c][[b, c], a^b] = 1$ where a, b, c are elements of some group G. You will also need to use of HW#8.2 along with the observation that $a^b = a[a, b]$ for all $a, b \in G$ to prove (2),(3) and (4).

   Warning: You need to prove (2), (3) and (4) for all elements of $L(G)$, not just homogeneous ones. The reduction to the homogeneous case is straighforward for (2) and (4), but not for (3).

3. Given $n \geq 2$ and an ordered n-tuple $g_1, \ldots, g_n$ of elements of a group G, the *left-normed commutator* $[g_1, \ldots, g_n]$ is defined as follows. If $n = 2$, this is the usual commutator, and for $n \geq 3$ we define $[g_1, \ldots, g_n]$ inductively by $[g_1, \ldots, g_n] = [[g_1, \ldots, g_{n-1}], g_n]$.

(a) Let G be a group. Prove that $\gamma_n G$ (the n^{th} term of the lower central series of G) is generated by all left-normed commutators of length n.

(b) Again let G be a group, and let $L(G)$ be the Lie ring associated to the lower central series of G (that is, we set $G_n = \gamma_n G$ in the definition of $L(G)$). Use (a) to prove that $L(G)$ is generated as a Lie ring by $L_1(G) = G_1/G_2$.

4. Recall from class that a group G is *residually finite* if it satisfies the following equivalent conditions:

(1) For every $1 \neq g \in G$ there exists a finite index subgroup N of G such that $g \notin N$.

(2) For every $1 \neq g \in G$ there exists a finite index normal subgroup N of G such that $g \notin N$.

(3) For every $1 \neq g \in G$ there exists a finite group P and a homomorphism $\varphi : G \to P$ such that $\varphi(g) \neq 1$.

(3') For every two distinct elements $x, y \in G$ there exists a finite group P and a homomorphism $\varphi : G \to P$ such that $\varphi(x) \neq \varphi(y)$.

(4) G is a subgroup of a direct product of some family of finite groups.

Now the actual problem.

(a) Prove that the above conditions are indeed equivalent.

(b) Suppose we are given a short exact sequence of groups $1 \to K \to G \to Q \to 1$ where K is residually finite and Q is finite. Prove that G is residually finite.

(c) Again assume that $1 \to K \to G \to Q \to 1$ is exact, but now K is finite and Q is residually finite. Is G residually finite? Prove or give a counterexample.