Math 8851. Homework #8. To be completed by Thu, Apr 6

1. Recall the statement of Gromov's polynomial growth theorem.

Theorem (Gromov)**.** *Let G be a finitely generated group of polynomial growth (that is, $b_G(n) \preceq n^d$ for some $d \in \mathbb{N}$). Then G is virtually nilpotent.*

Deduce Gromov's theorem from the following result:

Theorem (Gromov's virtual fibering theorem)**.** *Let G be a finitely generated group of polynomial growth. Then G has a finite index subgroup G_1 which admits an epimorphism onto $\mathbb{Z}$.*

Hint: Let $\pi : G_1 \to \mathbb{Z}$ be the epimorphism from the virtual fibering theorem and let $H = \mathrm{Ker}\,(\pi)$. First show that H is finitely generated (this follows immediately from one of the theorems from class). Then use a direct counting argument to show that $b_G(n) \succeq b_H(n) \cdot n$. Finally use this inequality and the Milnor-Wolf theorem to prove Gromov's theorem by induction on d.

2. Let G be a group.

(a) Prove the following commutator identities (here x, y, z are arbitrary elements of G):

(i) $[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z]$.

(ii) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$.

(b) Use (a) to show that if N and M are normal subgroups of G, S is a subset of N which generates N as a normal subgroup and T is a subset of M which generates M as a normal subgroup, then $[N, M]$ is normally generated by the set $\{[s, t] : s \in S, t \in T\}$.

3. The goal of this problem is to prove Lemma 20.2 from class (the statement is recalled below). Let $G = \mathbb{Z} \ltimes_A \mathbb{Z}^n$ where $A \in \mathrm{GL}_n(\mathbb{Z})$. Recall that by definition $\mathbb{Z} \ltimes_A \mathbb{Z}^n$ is the group $\mathbb{Z} \ltimes_\varphi \mathbb{Z}^n$ where $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ is the unique homomorphism such that $\varphi(1) = A$.

As in class, we will write G as an internal semidirect product $G = \langle x \rangle \ltimes V$ where $V \cong \mathbb{Z}^n$ and $x^{-1}vx = Av$ for all $v \in V$. Also recall that we are switching to additive notation inside V.

Prove Lemma 20.2 which asserts that for all $k \geq 2$

$$\gamma_k G = (A - 1)^{k-1} V.$$

Hint: Induction on k. First use Problem 2 to prove that $(A-1)^{k-1}V$ generates $\gamma_k G$ as a normal subgroup. Then show that $(A-1)^{k-1}V$ is already a normal subgroup of G, thus finishing the proof.

4. Consider the p-lamplighter group $G_p = \mathbb{Z} wr (\mathbb{Z}/p\mathbb{Z})$.

(a) Use the proof of Proposition 19.2 from class to exhibit specific elements $a, b \in G_p$ which generate a free subsemigroup.

(b) An HNN extension $G = \langle H, t \mid t^{-1}at = \varphi(a) \text{ for all } a \in A\rangle$ is called *ascending* if $\varphi(A) \subseteq A$ and *propertly ascending* if $\varphi(A)$ is a proper subgroup of A. Prove that G_p can be realized as a propertly ascending HNN extension. **Hint:** Use the presentation from HW 2.4.

Note: One can prove that every ascending HNN extension contains a free subsemigroup arguing very similarly to the corresponding part of the proof of Proposition 19.2. One can also show that if $\pi : G \to \mathbb{Z}$ is an epimorphism such that G is finitely generated and $H = \operatorname{Ker} \pi$ is not finitely generated, then H can be represented as an ascending HNN extension. This provides a more conceptual interpretation of the proof of Proposition 19.2 given in class.