Math 8851. Homework #7. To be completed by Thu, Mar 30

1. Let Ω be the set of all functions $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ which are non-decreasing (that is, $f(n) \leq f(m)$ whenever $n \leq m$) – note that any growth function $b_{G,S}$ lies in Ω. It is not hard to show that the restriction of the relation $\preceq$ to Ω can be defined by the following simpler condition (this is not part of the problem):

$$f \preceq g \iff \text{there exists } C \in \mathbb{N} \text{ such that } f(x) \leq Cg(Cx) \text{ for all } n \in \mathbb{N}.$$

As before, define $f \sim g$ for $f, g \in \Omega$ if $f \preceq g$ and $g \preceq f$.

Prove that the relation $\preceq$ on Ω is transitive and that $\sim$ is an equivalence relation.

2. Let S_1 and S_2 be finite generating sets for the same group G. Prove that $b_{G,S_1} \sim b_{G,S_2}$. Recall that

$$B_{G,S}(n) = \{g \in G : l_S(g) \leq n\} \text{ and } b_{G,S}(n) = |B_{G,S}(n)|.$$

3. Let H be a subgroup of a group G.

(a) Prove that $b_H \preceq b_G$. Recall that for a group Γ, b_Γ is the equivalece class of the functions $b_{\Gamma,S}$ where S is a finite generating set for Γ (all such functions are indeed equivalent by Problem 2).

(b) Prove that if H has finite index in G, then $b_H \sim b_G$.

4. Let G be an infinite group and S a finite generating set for G. Prove that $b_{G,S}(n) \geq n$ for all n.

5. Let G be a group and S a finite generating set for G. As before, for $n \in \mathbb{Z}_{\geq 0}$ let $\Sigma_{G,S}(n) = \{g \in G : l_S(g) = n\}$ (the sphere of radius n) and $\sigma_{G,S}(n) = |\Sigma_{G,S}(n)|$. The formal power series $f_{G,S}(t) = \sum_{n=0}^{\infty} \sigma_{G,S}(n)t^n$ is called the *spherical growth series* of G with respect to S.

(1) Suppose $G = \langle S \rangle$ and $H = \langle T \rangle$ where S, T are finite. Prove that

$$f_{G\times H, S\cup T}(t) = f_{G,S}(t) \cdot f_{H,T}(t).$$

(2) Let G, H, S and T be as in (1). Prove that

$$f_{G*H,S\cup T}(t) = \frac{f_{G,S}(t) \cdot f_{H,T}(t)}{f_{G,S}(t) + f_{H,T}(t) - f_{G,S}(t) \cdot f_{H,T}(t)}.$$

Hint: Let $a = a(t) = f_{G,S}(t) - 1$ and $b = b(t) = f_{H,T}(t) - 1$. First argue that $f_{G*H,S\cup T}(t) = 1 + a + b + ab + ba + aba +$

$bab + \dots$ (this follows from the normal form of elements of a free product).

(3) Let $G = \mathbb{Z}^d$ and $S = \{e_1, \dots, e_d\}$ (the standard basis). Use (1) to compute precisely $f_{G,S}(t)$ and $\sigma_{G,S}(n)$.

(4) Now let $X = \{x_1, \dots, x_d\}$ and $F = F(X)$. Prove that

$$f_{G,X}(t) = \frac{1+t}{1-(2d-1)t}$$

in 2 different ways: using the formula $\sigma_{G,S}(n) = 2d(2d-1)^n$ for $n \geq 1$ derived in class and then using (2).