Math 8851. Homework #6. To be completed by Thu, Mar 16

1. Give an example of a group G acting on an **unoriented** tree T with edge inversions such that

(i) the quotient graph $G\backslash T$ is a loop
(ii) G cannot be decomposed as an HNN extension.

Remark. Note that a group cannot act on an oriented tree T with edge inversions (since in that case T contains edges (v, w) and (w, v) for some vertices v and w and thus T is not a tree). Usually when one is taking about a group acting on a tree, it is assumed that the tree is unoriented. In class we implicitly assumed that groups acts on oriented trees; we did not lose any generality by doing so since we always required that the group acts without edge inversions. Indeed, if G acts on an unoriented tree T without edge inversions, it is not hard to show that one can always choose orientation on T which is presereved under the action (pick one edge in every orbit of the action of G on $E(T)$ and orient those edges arbitrarily; then there is a unique way to orient the remaining edges so that the orientation is preserved by the G-action).

2. Recall the formula for the rank of the Cartesian subgroup of the free product of a finite family of finite groups from HW#5.3(b):

If $G_1, \ldots, G_k$ are finite groups and C is the Cartesian subgroup of $*G_i$, that is, the kernel of the natural map $*G_i \to \prod G_i$, then

$$\mathrm{rk}(C) = \prod_{i=1}^{k} |G_i| \left(k - 1 - \sum_{i=1}^{k} \frac{1}{|G_i|} \right) + 1.$$

Give another proof of this formula, this time using the Bass-Serre theory.

Hint: Let $G = *G_i$, realize G as $\pi_1(\mathbb{G}, Y)$ for some graph of groups $(\mathbb{G}, Y)$, as in the proof of the Kurosh Subgroup Theorem (KST), and consider the action of G on the associated Bass-Serre tree X. The proof of KST shows that to compute the rank of C it suffices to know the numbers of vertices and edges in the quotient graph $C\backslash X$. To compute these first prove the following lemma.

Lemma. *If G is a group, K is a finite subgroup of G and H is a finite index normal subgroup of G such that $K \cap H = \{1\}$, then the number of orbits of the left-multiplication action of H on G/K is $\frac{[G:H]}{|K|}$.*

3. Before doing this problem read about residually finite and hopfian groups in online Lecture 6. Prove Proposition 6.3 from online notes which asserts that a finitely generated residually finite group must be hopfian. **Hint:** Let G be finitely generated and $\varphi : G \to G$ an epimorphism. Then $G/\mathrm{Ker}\,\varphi \cong G$. Deduce that for any $n \in \mathbb{N}$ there is a bijection between the set of all normal subgroups of index n in G and the set of those normal sugbroups of index n in G which contain $\mathrm{Ker}\,\varphi$. Then use HW#5.4 to show that $\mathrm{Ker}\,\varphi$ must lie in the intersection of all finite index subgroups of G.

4. Given natural numbers m and n, the Baumslag-Soliar group $BS(m,n)$ is defined by

$$BS(m,n) = \langle t, s \mid t^{-1}s^m t = s^n \rangle.$$

(a) Show that $BS(m,n)$ is an HNN-extension.
(b) Prove that there exists an epimorphism $\varphi : BS(m,n) \to BS(m,n)$ such that $\varphi(t) = t$ and $\varphi(s) = s^2$.
(c) Now assume that $m = 2$ and $n = 3$. Show that the elements s and $s^t = t^{-1}st$ do not commute (use Britton's Lemma) while their images under φ from (b) commute. Deduce that $BS(2,3)$ is not Hopfian.