Math 8851. Homework #4. To be completed by Thu, Feb 23

1. Prove that for any integer $n \geq 2$ the matrices $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ generate a free group of rank two. **Hint:** Consider the natural action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$ and apply the Ping-Pong Lemma with suitable subsets X_1 and X_2 (there exist X_1 and X_2 which work for every $n \geq 2$).

2. Use the isomorphism $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ established in Lecture 9 to prove that

$$SL_2(\mathbb{Z}) = \langle A, B \mid A^4 = 1, A^2 = (AB)^3 \rangle$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. **Hint:** Show that in the group given by the presentation $\langle a, b \mid a^4 = 1, a^2 = (ab)^3 \rangle$ the element a^2 is central and has order 2.

Deduce that $SL_2(\mathbb{Z})$ decomposes as the amalgam $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$.

3. The goal of this problem is to give another proof of the equality $\mathrm{IA}_2 = \mathrm{Inn}(F_2)$ using the result of Problem 2.

Define $\mathrm{Out}^+(F_2) = \mathrm{Aut}^+(F_2)/\mathrm{Inn}(F_2)$. Since $\mathrm{Aut}^+(F_2)$ is generated by the Nielsen maps $R_{12}, R_{21}, L_{12}, L_{21}$ by HW#3.3, $\mathrm{Out}^+(F_2)$ is generated by their images in $\mathrm{Out}^+(F_2)$ which will denote by the same symbols with bars ($\overline{R_{12}}$ etc.).

(a) Prove that $\mathrm{Out}^+(F_2)$ is already generated by $\overline{R_{12}}$ and $\overline{R_{21}}$. Deduce that $\mathrm{Out}^+(F_2)$ is generated by $\overline{\alpha}$ and $\overline{\beta}$ where $\beta = R_{21}$ and α is given by $\alpha(x_1) = x_2, \alpha(x_2) = x_1^{-1}$.

(b) Verify by direct computation that $(\overline{\alpha})^4 = 1$ and $(\overline{\alpha})^2 = (\overline{\alpha}\overline{\beta})^3$. Deduce from Problem 2 that there exists an epimorphism $\varphi : SL_2(\mathbb{Z}) \to \mathrm{Out}^+(F_2)$ such that $\varphi(A) = \overline{\alpha}$ and $\varphi(B) = \overline{\beta}$.

(c) Since $\mathrm{Inn}(F_2) \subseteq \mathrm{IA}_2$, there is a natural projection map from $\mathrm{Out}^+(F_2) = \mathrm{Aut}^+(F_2)/\mathrm{Inn}(F_2)$ to $\mathrm{Aut}^+(F_2)/\mathrm{IA}_2$, and as explained in Lecture 8, $\mathrm{Aut}^+(F_2)/\mathrm{IA}_2$ is naturally isomorphic to $SL_2(\mathbb{Z})$. Thus we obtain an epimorphism $\pi : \mathrm{Out}^+(F_2) \to SL_2(\mathbb{Z})$. Check that $\pi(\overline{\alpha}) = A$ and $\pi(\overline{\beta}) = B$. Combining this with φ from (b), deduce that π is an isomorphism and therefore $\mathrm{IA}_2 = \mathrm{Inn}(F_2)$.

4. Complete the proof (started in Lecture 10) of the fact that the graph Γ associated to an amalgamated free product is a tree.

Recall the setup from class: $G = P *_A Q$. The vertex set of Γ is $V(\Gamma) = G/P \sqcup G/Q$, the edge set is $E(\Gamma) = G/A$, and for every edge gA its initial vertex $\alpha(gA)$ and end vertex $\omega(gA)$ are $\alpha(gA) = gP$ and $\omega(gA) = gQ$.

We observed that any cycle in Γ must have even length, say $2k$. Let $v_0, v_1, \ldots, v_{2k} = v_0$ by the vertices in the cycle. WOLOG $v_0 = gP$ for some $g \in G$. In class we argued that there exist elements $p_1, \ldots, p_k \in P$ and $q_1, \ldots, q_k \in Q$ such that $v_{2m} = g \prod_{i=1}^{m} p_i q_i P$ and $v_{2m-1} = g(\prod_{i=1}^{m-1} p_i q_i) p_m Q$ for all $1 \leq m \leq k$. We deduced that $\prod_{i=1}^{k} p_i q_i = p$ for some $p \in P$. If none of the elements p_i, q_i lie in A, we obtained a contradiction with the uniqueness of normal forms.

To finish the proof it remains to show that if some p_i or q_i lies in A, then the cycle must have backtracking (that is, two consecutive edges are inverse to each other). This is what you need to prove in this problem. Most likely, you will first need to show that Γ does not have multiple edges.