Math 8851. Homework #3. To be completed by Thu, Feb 16

1. Let $G = A \ltimes_{\varphi} B$ be the (external) semidirect product of groups A and B corresponding to some homomorphism $\varphi : A \to \mathrm{Aut}(B)$. Suppose we are given presentations by generators and relations for both A and B:

$$A = \langle X_1 | R_1 \rangle \qquad B = \langle X_2 | R_2 \rangle.$$

Find (with proof) a presentation for G in terms of X_1, X_2, R_1, R_2 and φ.

Note: If you succeeded in proving Hall's Theorem asserting that an extension of finitely presented groups is finitely presented (HW# 2.2), this problem should be straightforward. If you did not succeed in solving HW# 2.2, you may want to start with this problem and then come back to HW# 2.2.

2. Let p be a prime. Prove that the lamplighter group $G_p = \mathbb{Z}\, wr\, \mathbb{Z}/p\mathbb{Z}$ is not finitely presented.

Hint: As discussed in Lecture 5, if G is a finitely presented group, then for any generating set X of G there exists a finite presentation $G = \langle X | R \rangle$. Moreover, if we already have some presentation $\langle X | R_0 \rangle$ for G, then one can assume that R is a finite subset of R_0 (do you see why?)

Suppose now that G_p is finitely presented. Then by the previous paragraph and HW# 2.4(c) there exists $N \in \mathbb{N}$ such that G_p admits the following (finite) presentation:

$$\langle x, y \mid y^p = 1, [y, y^x] = 1, [y, y^{x^2}] = 1, [y, y^{x^{N-1}}] = 1, \ldots \rangle \qquad (***)$$

and such that the relation $[y, y^{x^i}] = 1$ holds in G_p for all $i \in \mathbb{Z}_{\geq 0}$.

Now prove that this is impossible as follows. Show that for sufficiently large M (depending on N) the symmetric group S_M contains 2 elements t and s such that $t^p = 1$ and $[t, t^{s^i}] = 1$ for all $1 \leq i \leq N-1$, but $[t, t^{s^N}] \neq 1$. Then apply von Dyck's theorem to get a contradiction.

3. Recall from class that $\mathrm{Aut}^+(F_n)$ is the preimage of $SL_n(\mathbb{Z})$ under the natural "abelianization" map $\pi : \mathrm{Aut}(F_n) \to GL_n(\mathbb{Z})$ (which is surjective, as proved in Lecture 8) and thus $[\mathrm{Aut}(F_n) : \mathrm{Aut}^+(F_n)] = 2$. The goal of this problem is to prove that $\mathrm{Aut}^+(F_n)$ is generated by the elements R_{ij} and L_{ij} (recall that R_{ij} sends x_i to $x_i x_j$ and fixes all other x_k and L_{ij} sends x_i to $x_j x_i$ and fixes all other x_k).

Define $H = \langle R_{ij}, L_{ij} \rangle$. Then $H \subseteq \mathrm{Aut}^+(F_n)$, and to prove the equality it suffices to show that $[\mathrm{Aut}(F_n) : H] = 2$.

(a) Recall that $\mathrm{Aut}(F_n)$ is generated by the elements R_{ij}, L_{ij}, I_i and P_σ, with $\sigma \in S_n$, where I_i inverts x_i and fixes all other x_k and P_σ sends x_k to $x_{\sigma(k)}$ for all k. Use this fact to prove that H is normal in $\mathrm{Aut}(F_n)$.

(b) For any $1 \leq i \neq j \leq n$ let Q_{ij} be the element of $\mathrm{Aut}(F_n)$ given by $x_i \mapsto x_j$, $x_j \mapsto x_i^{-1}$ and $x_k \to x_k$ for $k \neq i, j$. Prove by direct computation that $Q_{ij} \in \mathrm{Aut}^+(F_n)$.

(c) Given $g \in \mathrm{Aut}(F_n)$, let $\overline{g}$ denote the image of g in $\mathrm{Aut}(F_n)/H$. Use (b) to show that $\overline{P_{(ij)}} = \overline{I_i}$ for any $i \neq j$ (here (ij) is the transposition swapping i and j). Deduce from this that $|\mathrm{Aut}(F_n)/H| = 2$ and thus $[\mathrm{Aut}(F_n) : H] = 2$.

4. Recall from class that IA_n (also called the Torelli subgroup of $\mathrm{Aut}(F_n)$) is the kernel of $\pi : \mathrm{Aut}(F_n) \to GL_n(\mathbb{Z})$.

(a) Prove that IA_n contains $\mathrm{Inn}(F_n)$, the subgroup of inner automorphsisms of F_n.

(b) Magnus (1935) proved that IA_n is generated by the elements K_{ij} with $1 \leq i \neq j \leq n$ and K_{ijm} with i, j, m distinct where K_{ij} sends x_i to $x_j^{-1} x_i x_j$ and fixes all other x_k and K_{ijm} sends x_i to $x_i[x_j, x_m]$ and fixes all other x_k. Verify that the elements K_{ij} and K_{ijm} indeed lie in IA_n.

(c) Use (b) to show that $\mathrm{IA}_2 = \mathrm{Inn}(F_2)$. We will discuss a different proof of this result later in the course.

5. The proof of the Nielsen reduction theorem (Theorem 7.5) yields a general algorithm which, given an n-tuple of elements of F_n, decides whether these elements generate F_n or not. In the case $n = 2$ one can answer this question almost immediately using the following commutator test.

Theorem (Commutator test)**.** *Let $\{x, y\}$ be a free generating set of F_2, and take any $u, v \in F_2$. Then u and v generate F_2 if and only if the commutator $[u, v] = u^{-1}v^{-1}uv$ is conjugate (in F_2) to $[x, y]$ or $[y, x] = [x, y]^{-1}$.*

(a) Prove the 'only if' ($\Rightarrow$) part of the commutator test. **Hint:** Use Nielsen moves.

(b) Now think of how you would prove the ‘if’ part. I do not know of a nice short algebraic argument. One possible proof is outlined in the following paper of Shpilrain (see Proposition 2.4):

`https://shpilrain.ccny.cuny.edu/test1.pdf`