Math 8851. Homework #2. To be completed by Thu, Feb 9

1. Let G be a finitely presented group. Define the deficiency of G, denoted by $def(G)$, to be the supremum of the quantity $|X|-|R|$ where (X, R) runs over all presentations of G by generators and relators.

(a) Let H be a subgroup of G of index n. Prove that $def(H) \geq 1 + (def(G) - 1) \cdot n$.

(b) Prove that $def(G) \leq d(G^{ab})$ where $G^{ab} = G/[G,G]$ is the abelianization of G and $d(\cdot)$ denoted the minimal number of generators. In particular, this implies that $def(G) < \infty$.

Hint: Suppose that $G = \langle X|R\rangle$ where $X = \{x_1, \ldots, x_n\}$ and $R = \{r_1, \ldots, r_m\}$. Show that G^{ab} is isomorphic to $\mathbb{Z}^n/M$ where M is a subgroup of $\mathbb{Z}^n$ generated by m elements.

2. The goal of this problem is to fill in the details of the proof of Hall's theorem (extensions of finitely presented groups are finitely presented). So, suppose we are given a short exact sequence of groups

$$1 \to N \to G \xrightarrow{\pi} Q \to 1$$

where N and Q are finitely presented. Our goal is to prove that G is finitely presented.

As in class, we can assume that $Q = G/N$ and $\pi(g) = gN$ for all $g \in G$. Start by choosing finite presentations $N = \langle Y|A\rangle$ and $Q = \langle Z|B\rangle$ where $Y = \{y_1, \ldots, y_k\}$ and $Z = \{z_1, \ldots, z_m\}$. For each $1 \leq i \leq m$ choose $\widetilde{z}_i \in G$ such that $\pi(\widetilde{z}_i) = z_i$, and let $\widetilde{Z} = \{\widetilde{z}_1, \ldots, \widetilde{z}_m\}$. We claim that $Y \cup \widetilde{Z}$ is a generating set of G.

Now take any $b \in B$ and think of b as a word in $z_1, \ldots, z_m$. Let $\widetilde{b} = b(\widetilde{z}_1, \ldots, \widetilde{z}_m)$ be the word obtained from b by replacing each z_i by $\widetilde{z}_i$, and let $\widetilde{b}_{\widetilde{G}}$ be the element of $\widetilde{G}$ represented by $\widetilde{b}$. Then $\pi(\widetilde{b}_{\widetilde{G}}) = b(z_1, \ldots, z_n) = 1$ in G, so $\widetilde{b}_{\widetilde{G}} \in N$. Since N is generated by $y_1, \ldots, y_k$, there exists a word w_b in $y_1, \ldots, y_k$, such that the word $\widetilde{b} \cdot w_b(y_1, \ldots, y_k)^{-1}$ is a relator of G (with respect to the generating set $Y \cup \widetilde{Z}$). Let

$$\widetilde{B} = \{\widetilde{b} \cdot w_b(y_1, \ldots, y_k)^{-1} : b \in B\}.$$

Next for each $1 \leq i \leq m$ and $1 \leq j \leq k$ the element $\widetilde{z}_i y_j \widetilde{z}_i^{-1}$ lies in N (since N is normal in G), so $\widetilde{z}_i y_j \widetilde{z}_i^{-1} = w_{ij}(y_1, \ldots, y_k)$ for some word w_{ij} in $y_1, \ldots, y_k$. Let

$$C = \{\widetilde{z}_i y_j \widetilde{z}_i^{-1} \cdot w_{ij}(y_1, \ldots, y_k)^{-1} : 1 \leq i \leq m, 1 \leq j \leq k\},$$

and let $\widetilde{G}$ be the group given by the presentation $\langle Y \cup \widetilde{Z} | R \rangle$ where $R = A \cup \widetilde{B} \cup C$. By construction each element of R is a relator of G, so by von Dyck's theorem there exists a homomorphism $\varphi : \widetilde{G} \to G$ which sends every $x \in Y \cup \widetilde{Z}$ considered as an element of $\widetilde{G}$ to the same x considered as an element of G. Our goal is to show that φ is an isomorphism.

Now the actual problem starts.

(a) Prove that $Y \cup \widetilde{Z}$ is indeed a generating set for G. Deduce that φ is surjective.

(b) Now let $\widetilde{N}$ be the subgroup of $\widetilde{G}$ generated by Y. Show that $\widetilde{N}$ is normal in $\widetilde{G}$ and φ maps $\widetilde{N}$ isomorphically onto N (to prove that φ restricted to $\widetilde{N}$ is an isomoprphism use von Dyck's theorem to construct a natural map $\psi : N \to \widetilde{N}$ and show that it is inverse to φ restricted to N).

(c) Deduce from (b) that φ induces a natural map $\overline{\varphi} : \widetilde{G}/\widetilde{N} \to G/N$ and then prove that $\overline{\varphi}$ is an isomorphism similarly to (b).

(d) Finally deduce from (b) and (c) that $\varphi : \widetilde{G} \to G$ is an isomorphism.

3. Consider the following condition (*) on a group G:

(*) There exists a central extension $1 \to N \to \widehat{G} \to G \to 1$ (that is, this sequence is exact and N lies in $Z(\widehat{G})$, the center of $\widehat{G}$) such that $\widehat{G}$ is finitely generated and N is not finitely generated.

In Lecture 5 we proved that any group satisfying (*) is not finitely presented (note that such group is obviously finitely generated being a quotient of the finitely generated group $\widehat{G}$).

(a) Prove that G satisfies (*) if and only if $G \cong F/N$ for some finitely generated group F and a normal subgroup N such that the quotient $N/[N, F]$ is infinitely generated (if A and B are subgroups of the same group, $[A, B]$ is the subgroup generated by all commutators $[a, b]$ with $a \in A$, $b \in B$).

(b) Let G be a finitely generated group, so that $G \cong F/N$ for some finitely generated group F and a normal subgroup N. For simplicity of notation we will assume below that $G = F/N$.

Prove that each of the following conditions implies the next one:

(i) G is finitely presented

(ii) $N^{ab} = N/[N, N]$ is finitely generated as a G-module where $G = F/N$ acts on N^{ab} by conjugation:

$$(gN).(y[N, N]) = gyg^{-1}[N, N] \text{ for all } g \in G, y \in N.$$

(iii) $N/[N, F]$ is finitely generated as a group.

Using the Hopf formula mentioned in class, it is not hard to show that (iii) holds if and only if $H_2(G, \mathbb{Z})$ is finitely generated. Once can also show that (ii) holds if and only if G has property (FP_2).

4. For any groups A and B their *wreath product* $A\, wr\, B$ is defined as follows. Let $C = (\oplus_{a\in A} B)$, and think of elements of C as functions $f : A \to B$ with finite support (that is, $f(a) = 1_B$ for all but finitely many $a \in A$). Define the homomorphism $\varphi : A \to \mathrm{Aut}(C)$ by

$$(\varphi(x)f)(a) = f(x^{-1}a) \text{ for all } x, a \in A \text{ and } f \in C.$$

The wreath product $A\, wr\, B$ is the semidirect product $A \ltimes_\varphi C$.

As usual, we can think of A and C as subgroups of $A\, wr\, B$. For each $a \in A$ and $b \in B$ let b_a be the element of C defined by $b_a(a) = b$ and $b_a(x) = 1_B$ for all $x \in B \setminus \{a\}$.

(a) Verify that $xb_ax^{-1} = b_{xa}$ for all $x, a \in A$ and $b \in B$.

(b) Let S be a generating set for A and T a generating set for B. Given $a \in A$, let $B_a = \{b_a : b \in B\}$ – this is a subgroup of C isomorphic to B; you should think of it as a copy of B indexed by a – and let $T_a \subseteq B_a$ be the corresponding copy of T. Prove that if we fix $x \in A$, then $T_x \cup S$ is a generating set for $A\, wr\, B$. Deduce that the wreath product of finitely generated groups is finitely generated. **Hint:** First show that $\cup_{a\in A} T_a \cup S$ is a generating set for $A\, wr\, B$.

(c) Given a prime p, the group $G_p = \mathbb{Z}\, wr\, \mathbb{Z}/p\mathbb{Z}$ is called the p-lamplighter group. Prove that G_p has the following presentation by generators and relations:

$$\langle x, y \mid y^p = 1, [y, y^x] = 1, [y, y^{x^2}] = 1, [y, y^{x^3}] = 1, \dots \rangle.$$

Here $g^h = h^{-1}gh$.