Math 8851. Homework #10. To be completed by Thu, Apr 20

1. Let p be a prime. As stated in class, if G is any group such that $g^p = 1$ for all $g \in G$ and $L(G)$ is the graded Lie ring associated to the lower central series of G, then $L(G)$ is a Lie algebra over $\mathbb{F}_p$ satisfying the Engel identity (E_{p-1}):

$$[y, \underbrace{x, \ldots, x}_{p-1 \text{ times}}] \text{ for all } x, y \in L(G) \qquad (E_{p-1})$$

where the bracket is left-normed.

The goal of this problem is prove an analogous result for associative algebras. Let A be an associative algebra (without 1) satisfying the identity $a^p = 0$ for all $a \in A$. Prove that considered as a Lie algebra (with the bracket $[x, y] = xy - yx$), A satisfies (E_{p-1}). **Hint:** Consider the identity $(x + \lambda y)^p = 0$ with $x, y \in A$ and $\lambda \in \mathbb{F}_p$ and compute the coefficient of λ.

2. Let L be a Lie algebra over $\mathbb{F}_3$ satisfying the Engel identity (E_2), that is, $[y, x, x] = 0$ for all $x, y \in L$. Prove that $\gamma_4 L = 0$ (that is, L is nilpotent of class at most 3). Moreover, show that if X generates L as a Lie ring and we choose a total order on X, then L is spanned by left-normed commutators of length ≤ 3 where all the entries are elements of X and appear in decreasing order (that is, X itself, $[x, y]$ with $x > y$ and $[x, y, z]$ with $x > y > z$).

Deduce (using the results from class and HW#9) that if G is any group generated by d elements and satisfying $g^3 = 1$ for all $g \in G$, then $|G| \leq 3^{d + \binom{d}{2} + \binom{d}{3}}$.

You may use without proof that if an arbitrary Lie ring L is generated by a set X as a Lie ring, then left-normed commutators of elements of X span L (as an additive group). Here is a proposed sketch of the proof (all brackets below are left-normed):

(a) Replacing x by $x_1 + x_2$ in (E_2) and using the Jacobi identity, show that for any $x_1, x_2, x_3 \in L$ and any $\sigma \in S_3$ (the symmetric group on $\{1, 2, 3\}$) we have the identity

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = (-1)^{\sigma} [x_1, x_2, x_3].$$

(b) Now use (a) (and (E_2) and Jacobi identity again) to show that $[x, y, z, w] = 0$ whenever there is a repetition in the sequence x, y, z, w.

(c) Now starting with the result of (b) and applying the same trick as in (a), show that

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] = (-1)^{\sigma}[x_1, x_2, x_3, x_4]$$

for all $x_1, x_2, x_3, x_4 \in L$ and $\sigma \in S_4$.

(d) Finally, use (c) and the Lie algebra axioms again to show that all left-normed commutators of length 4 are equal to 0. The moreover part of the assertion of the problem should follow from what you already established in (a).

3. Fix a prime p and an integer d, and define the sequence $\{f_k(d,p)\}_{k=1}^{\infty}$ by $f_1(d,p) = d$ and $f_k(d,p) = 1 + (f_{k-1}(d,p) - 1)p^{f_{k-1}(d,p)}$ for $k \geq 2$. Prove that for all $k \in \mathbb{N}$ there exists a d-generated group $G_k(d,p)$ such that $g^{p^k} = 1$ for all $g \in G_k(d,p)$ and $|G_k(d,p)| = f_k(d,p)$ (this provides a lower bound for the order of the restricted Burnside group $R(d, p^k)$). **Hint:** Let F_d be a free group of rank d. Construct a suitable sequence of normal subgroups $F_d = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_k$ such that N_i/N_{i+1} has exponent p and define $G_k(d,p) = F_d/N_k$.

Note: The sequence $f_k(d,p)$ grows very fast. In particular, $f_k(d,p) \geq p^{p^{p^{\cdots^{p^d}}}}$ where the total number of exponentiations is k. There is a known upper bound for $R(d, p^k)$ of the same form (iterated exponentials), but the number of exponentiations is considerably larger.