Math 8851. Homework #3. To be completed by 5pm on Fri, Oct 6

Below [DDMS] refers to the book 'Analytic pro-p groups', 2nd edition by Dixon, du Sautoy, Mann and Segal.

1. This is a carryover from HW#2, namely parts (b) and (c) of HW#2.5. Note that there are new hints in both (b) and (c).

(b) Let G be a pro-p group which is not finitely generated (as usual topologically). Prove that there exists a closed normal subgroup K of G such that $G/K \cong \mathbb{F}_p^\infty$. **Hint:** Use Proposition 1.13 from [DDMS] and the fact that every abelian pro-p group of exponent p is isomorphism to $\mathbb{F}_p^I = \prod\limits_{i\in I} \mathbb{F}_p$ for some set I (this appears, e.g. as Theorem 5.7 in Wilson's book 'Profinite groups'). Deduce from HW#2.5(a) that G has a finite index subgroup which is not open.

(c) Let $\{F_i\}_{i\in\mathbb{N}}$ be a family of finite groups of pairwise coprime orders such that $\{d(F_i)\}$ is unbounded (recall that $d(\cdot)$ denotes the minimal number of generators). Prove that the profinite group $G = \prod\limits_{i\in\mathbb{N}} F_i$ is not finitely generated, but every finite index subgroup of G is open. **Hint:** Use one of the tricks from the proof of Lemma 1.18 in [DDMS]. Also keep in mind that a finite index subgroup of a topological group is open if and only if it is closed.

2. Problem 1.18(i) from [DDMS] (page 34). This is another carryover from HW#2 (no changes in this problem)

3. Let X be an infinite set, $F(X)$ the free abstract group on X and Λ the set of all finite index normal subgroups N of $F(X)$ such that N contains all but finitely many elements of X. The group $\widehat{F(X)}_\Lambda$ (the completion of $F(X)$ with respect to Λ) is called the free profinite group on X.

(a) Prove that $|\Lambda| = |X|$. Deduce from HW#2 that the set of open subgroups of $\widehat{F(X)}_\Lambda$ has the same cardinality as X. In particular, if X is countable, $\widehat{F(X)}_\Lambda$ is countably based. **Note:** You may use without proof that $|X \times X| = |X|$ for any infinite set X.

(b) State and prove a natural universal property satisfied by $\widehat{F(X)}_{\Lambda}$ (it should be a minor variation of the usual universal property for finitely generated free profinite groups).

4. Let G be a finitely generated pro-p group and $d = d(G)$ its minimal number of generators.

(a) Prove that if X any (topological) generating set for G, then X contains a subset Y with $|Y| = d$ which generates G. **Hint:** Use the Frattini subgroup to reduce this problem to a basic fact from linear algebra.

(b) Give an example showing that (a) is false for abstract groups.

5. A topological group G is called *Hopfian* if every epimorphism $\phi : G \to G$ is an isomorphism.

(a) Prove that any finitely generated profinite group is Hopfian. **Hint:** use the fact that a finitely generated profinite group has finitely many subgroups of index n for any $n \in \mathbb{N}$ as well as a general relation between closed and open subgroups in profinite groups.

(b) Now let X be a finite set and $F = \widehat{F(X)}$, the profinite group on X. Let Y be another finite generating set of G. We say that Y is a *free generating set* for F if the unique homomorphism $\phi : \widehat{F(Y)} \to F$ such that $\phi_{|Y} : Y \to F$ is the inclusion map is an isomorphism. Prove that Y is a free generating set for F if and only if $|Y| = |X|$. The same is true if we replace free profinite groups by free pro-p groups.

6. The goal of this problem is to find an explicit profinite presentation for $\mathbb{Z}_p$. Recall that $\mathbb{Z}_p$ is a free pro-p group or rank 1, so it has a pro-p presentation $\langle x| \ \rangle$ (one generator and no relators). The same presentation in the category of profinite groups defines $\widehat{\mathbb{Z}}$. Since $\mathbb{Z}_p$ is procyclic, it still has a profinite presentation with 1 generator. Moreover, since any closed subgroup of a procyclic group is procyclic, $\mathbb{Z}_p$ has a profinite presentation with 1 generator and 1 relator. Finally, since every element of a procyclic group can be written as a profinite power of its generator, we deduce that $\mathbb{Z}_p$ has a profinite presentation $\langle x | x^{\alpha} \rangle$ for some $\alpha \in \widehat{\mathbb{Z}}$. Describe α explicitly (and prove your answer).

7. In Lecture 12 we proved that $d(G) = \dim H^1(G, \mathbb{F}_p)$ for any finitely generated pro-p group G (as before $d(G)$ is the minimal number of generators of G). Prove that the equality remains true even if G is infinitely generated (the equality should be interpreted as equality of cardinal numbers, not just $\infty = \infty$). **Note:** This is closely related to Problem 3.

8. Let G be a profinite group and A an abelian profinite group. As stated in class (Theorem 12.4) there is a natural bijection between the second cohomology group $H^2(G, A)$ (where we view A as a trivial G-module) and $Ext(G, A)$, the set of equivalence class of topological central extensions of G by A. This problem provides an outline of a proof of Theorem 12.4.

(a) Given $C \in H^2(G, A)$, let $Z : G \times G \to A$ be a 2-cocycle whose cohomology class is equal to C. Let E be the set of pairs $\{(g, a) : g \in G, a \in A\}$ with multiplication given by

$$(g_1, a_1) \cdot (g_2, a_2) = (g_1 g_2, a_1 + a_2 + Z(g_1, g_2)) \tag{1}$$

Let $\mathcal{E}$ be the sequence $(1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1)$ where where $\iota(a) = (1, a)$ and $\pi((g, a)) = g$ for any $a \in A$ and $g \in G$. Prove that $\mathcal{E}$ is a topological central extension and that its equivalence class depends only on C, not on Z.

(b) Conversely, let $\mathcal{E} = (1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1)$ be an element of $Ext(H, A)$. Let $\psi : G \to E$ be a continuous section of π, that is, a continuous map $G \to E$ such that $\pi \circ \psi = id_G$ (such a section exists since E and G are profinite – see, e.g. 1.3.3 in Wilson's book). Define $Z : G \times G \to A$ by

$$Z(g_1, g_2) = \iota^{-1}(\psi(g_1 g_2)^{-1} \psi(g_1) \psi(g_2)).$$

Prove that Z is a 2-cocycle whose cohomology class $[Z]$ is independent of the choice of ψ.

(c) Now prove that the maps $H^2(G, A) \to Ext(G, A)$ and $Ext(G, A) \to H^2(G, A)$ constructed in (a) and (b), respectively, are mutually inverse.