Math 8851. Homework #2. To be completed by 5pm on Fri, Sep 22

Below [DDMS] refers to the book 'Analytic pro-p groups', 2nd edition by Dixon, du Sautoy, Mann and Segal.

1. Problem 7(c) from HW#1. As suggested in the hint, first show that $\widehat{G}_\Lambda \cong \operatorname{proj\,lim}_{k\in\mathbb{N}} G/U_k$ (this reduces the problem to the case $\Lambda = \{U_k\}$). The latter is a special case of the following general statement:

Let $\{X_i\}_{i\in I}$ be an inverse system of sets or groups, and let J be a subset of I with the property that for every $i \in I$ there exists $j \in J$ with $j \geq i$. Then J is also a directed set, $\{X_j\}_{j\in J}$ is an inverse system (with the same transition maps) and $\operatorname{proj\,lim}_{j\in J} X_j \cong \operatorname{proj\,lim}_{i\in I} X_i$.

2. Let G be a profinite group and Λ a family of open normal subgroups closed under finite intersections with the property that $\bigcap_{N\in\Lambda} N = \{1\}$. Prove that Λ is a base of neighborhoods of 1 for G. **Hint:** Show that G is isomorphic to its Λ-completion $\widehat{G}_\Lambda$ as topological groups.

Note: By contrast, if G is an abstract group and Λ is a family of finite index normal subgroups closed under finite intersections with the property that $\bigcap_{N\in\Lambda} N = \{1\}$, then Λ need not form a base of neighborhoods of 1 for the profinite topology on G, that is, there may exist a finite index subgroup M of G which does not contain any element of Λ. Can you think of a specific example where this happens?

3. Let Γ be an abstract group, $G = \widehat{\Gamma}$ its profinite completion and $\iota : \Gamma \to G$ the canonical map. Prove that the map $N \mapsto \overline{\iota(N)}$ establishes a bijection between finite index subgroups of Γ and open subgroups of G.

Hint: First consider the case where ι is injective (in which case we can identify Γ with a subgroup of G) and then deduce the general case. The proof of Lemma 7.6 from class (given at the beginning of Lecture 8) should be helpful for this problem.

4. Let G be a non-trivial finite p-group. Prove that $[G, G]G^p$ is a proper subgroup of G without using the fact that $[G, G]G^p = \Phi(G)$ in this case. **Hint:** If $G = [G, G]G^p$ for some group G, the same is true for any quotient of G.

5.

(a) Let $\mathbb{F}_p^\infty = \prod_{k\in\mathbb{N}} \mathbb{F}_p$ be the product of countably many copies of $\mathbb{F}_p$ (here considered just as a cyclic group of order p). Prove that $\mathbb{F}_p^\infty$ has only countably many open subgroups, but uncountably many subgroups of index p and deduce that $\mathbb{F}_p^\infty$ has a finite index subgroup which is not open.

(b) Let G be a pro-p group which is not finitely generated (as usual topologically). Use Proposition 1.13 from the book and basic properties of Frattini subgroups to show that there exists a closed normal subgroup K of G such that $G/K \cong \mathbb{F}_p^\infty$. Deduce from (a) that G has a finite index subgroup which is not open.

(c) Let $\{F_i\}_{i\in\mathbb{N}}$ be a family of finite groups of pairwise coprime orders such that $\{d(F_i)\}$ is unbounded (recall that $d(\cdot)$ denotes the minimal number of generators). Prove that the profinite group $G = \prod\limits_{i\in\mathbb{N}} F_i$ is not finitely generated, but every finite index subgroup of G is open.

6. Problem 1.18(i) from [DDMS] (page 34).

Before the next problem we introduce a general definition.

Definition. Let R be a ring with 1 and I an ideal of R such that $\bigcap_{n\in\mathbb{N}} I^n = \{0\}$. Define the *$I$-adic norm* on R by $\|0\| = 0$, and for a nonzero x set $\|x\| = 2^{-\deg(x)}$ where $\deg(x) \in \mathbb{Z}_{\geq 0}$ is the unique integer such that $x \in I^{\deg(x)} \setminus I^{\deg(x)+1}$ (as usual we set $I^0 = R$).

The *I-adic metric* on R is defined by $d(x,y) = \|x-y\|$ (check that this is indeed a metric; in fact it is an ultra-metric). We say that R is *I-adically complete* if it is a complete metric space with respect to the I-adic metric (any R embeds into its I-adic completion $\widehat{R}_I$ which is always I-adically complete).

7. Let R and I be as in the above definition. Assume that

(a) R/I is a finite field. Let $p = char(R/I)$

(b) I is finitely generated as an ideal

(c) R is I-adically complete.

Prove that $G = 1 + I$ is a group and that it is a pro-p group with respect to the toopology induced from the I-adic topology on R.

One natural example of R and I satisfying (a)-(c) is $R = \mathbb{Z}_p$ (p-adics) and $I = p\mathbb{Z}_p$. Can you think of other examples?