Math 8851. Homework #1. To be completed by 5pm on Fri, Sep 8

Below [DDMS] refers to the book 'Analytic pro-p groups', 2nd edition by Dixon, du Sautoy, Mann and Segal.

1. Prove the implication (P2) $\Rightarrow$ (P1) in the statement about the equivalence of 4 definitions of a profinite group. That is, prove that if $G = \operatorname{proj}\lim_{i\in I} F_i$ for some inverse system of finite groups $\{F_i\}_{i\in I}$, then G is a closed subgroup of $\operatorname{proj}\lim_{i\in I} F_i$.

2. (Problem 3(ii) after Chapter 1 in [DDMS], page 31). Prove the universal property of the profinite completion: Let H be an abstract group, $\widehat{H}$ its profinite completion and $j : H \to \widehat{H}$ the canonical map. Prove that for any profinite group G and any homomorphism of abstract groups $\theta : H \to G$, there exists a unique continuous homomorphism $\widehat{\theta} : \widehat{H} \to G$ such that $\widehat{\theta} j = \theta$.

Hint: As is typical for a proof of a universal property, it is the existence part that requires more work. First prove this for finite G, in which case one can describe $\widehat{\theta}$ quite explicitly. Then use the universal property of inverse limits to extend the result to the general case.

3. Let G be a profinite group and p a prime number. Prove that the following 4 conditions on G are equivalent:

- (i) $G \cong \operatorname{proj}\lim_{i\in I} P_i$ for some inverse system of finite p-groups $\{P_i\}_{i\in I}$
- (ii) G has a base of neighborhoods of 1 consisting of open subgroups of p-power index
- (iii) Every open subgroup of G has p-power index
- (iv) Every continuous finite quotient of G is a p-group. Here by a continuous quotient of G we mean the image of a continuous homomorphism from G to another topological group, and finite groups are considered as discrete groups.

4. Let p be a prime, and let $\mathbb{Z}_p$ be the ring of p-adic integers, as defined in class, that is, $\mathbb{Z}_p = \operatorname{proj}\lim_{k\in\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}$. Prove that

- (a) $\mathbb{Z}_p$ is a local ring whose unique maximal ideal is $p\mathbb{Z}_p$;
- (b) Every nonzero ideal of $\mathbb{Z}_p$ is equal to $p^k\mathbb{Z}_p$ for some $k \in \mathbb{N}$ and $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$.

You may use the representation of $\mathbb{Z}_p$ as "power series in p" discussed in class.

5. Let p be a prime and $n \in \mathbb{N}$. Let $G = SL_n(\mathbb{Z}_p)$ and $H = SL_n^1(\mathbb{Z}_p)$, the first congruence subgroup of $SL_n(\mathbb{Z}_p)$, defined as the kernel of the natural projection map $SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p/p\mathbb{Z}_p)$, that is,

$$SL_n^1(\mathbb{Z}_p) = \{g \in G : g \equiv 1 \mod p\mathbb{Z}_p\}.$$

Prove that G is a profinite group and H is a pro-p group.

Note: The topology on G and H is induced from $Mat_n(\mathbb{Z}_p)$ where we identify $Mat_n(\mathbb{Z}_p)$ with $\mathbb{Z}_p^{n^2}$ endowed with the product topology.

6. Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$ considered as a ring. Prove the isomorphism

$$\mathbb{Z} \cong \prod_p \mathbb{Z}_p$$

where p ranges over all primes.

7. Let G be an abstract group and Λ a family of normal finite index subgroups of G closed under finite intersection. Recall that in this case there exists a topology on G (called the Λ-topology) which turns G into a topological group and where Λ is a base of neighborhoods of identity.

Assume now that the Λ-topology admits a countable base of neighborhoods of 1. Then there exists such a base $\{U_k\}_{k=1}^{\infty}$ where $U_1 \supseteq U_2 \supseteq \cdots$ and each $U_i \in \Lambda$. Define the pseudometric d on G as follows:

$$d(g,h) = \begin{cases} 0 & \text{if } g^{-1}h \in U_k \text{ for all } k \in \mathbb{N} \\ \frac{1}{k} & \text{if } g^{-1}h \notin U_k \text{ and } k \text{ is smallest with this property.} \end{cases}$$

(a) Prove that d is indeed a pseudo-metric.

(b) Let $\widetilde{G}$ be the completion of G with respect to d (the elements of $\widetilde{G}$ are equivalence classes of the Cauchy sequences of elements of G). Prove that if we define a binary operation $\cdot$ on $\widetilde{G}$ by $[x_n] \cdot [y_n] = [x_n y_n]$ (here $[a_n]$ is the equivalence class of a Cauchy sequence $\{a_n\}$), then $\cdot$ is well defined and $(\widetilde{G}, \cdot)$ is a topological group.

(c) Recall that the Λ-completion of G, denoted by $\widehat{G}_\Lambda$, is defined by $\widehat{G}_\Lambda = \operatorname{proj\,lim}_{U \in \Lambda} G/U$. Prove that $\widehat{G}_\Lambda \cong (\widetilde{G}, \cdot)$ as topological groups.

Hint: It is probably helpful to first show that $\widehat{G}_\Lambda \cong \operatorname{proj\,lim}_{k \in \mathbb{N}} G/U_k$.