Math 8851. Linear groups and Expander Graphs.

Suggested problems (through Lecture 11.)

Problem 1.1. Prove that if a finitely generated group G is linear over a field of characteristic zero, then it is actually linear over $\mathbb{C}$. (We will probably discuss this problem in class shortly).

Problem 1.2. Prove that the symmetric group S_n cannot be embedded in $GL_{n-2}(\mathbb{C})$.

Problem 1.3. Work out the details of the counterexample to Jordan's theorem in characteristic $p > 0$ described in Lecture 1.

Problem 4.1. Let R be a finitely generated domain of characteristic 0 and $d \geq 2$ an integer. As we proved in class, the group $GL_d(R)$ (and any of its subgroups) is virtually residually-p for every prime p outside certain finite set $B(R)$. For each of the rings $R = \mathbb{Z}$, $\mathbb{Z}[1/2]$ and $\mathbb{Z}[(1+\sqrt{-3})/2]$ do the following:

(a) Find the minimal possible set of bad primes $B(R)$, that is, the set of all primes p for which $GL_d(R)$ is not virtually residually-p.

(b) For each $p \notin B(R)$, give an explicit estimate on the index of a residually-p subgroup of $GL_d(R)$ which is guaranteed to be residually-p.

Problem 5.1. Recall Burnside's irreducibility criterion: if F is an algebraically closed field, then a subgroup $G \subseteq GL_n(F)$ is irreducible if and only if $FG = Mat_n(F)$, that is, G spans $Mat_n(F)$ as F-vector space. Deduce Burnside's irreducibility criterion from Schur's Lemma and the following theorem, called Jacobson's Density Theorem (it is directly related to the material discussed in Algebra-III, but may not have been stated there explicitly).

Jacobson's Density Theorem: *Let R be a ring with 1 and M an irreducible left R-module. Let $S = End_R(M)$, and note that M is naturally a left S-module. Prove that for any $a \in End_S(M)$ and any finite set of elements $m_1, \ldots, m_d \in M$ there exists $r \in R$ such that $am_i = rm_i$ for $1 \leq i \leq d$.*

Problem 5.2. Let F be a field which is not necessarily algebraically closed. Prove that Burnside's irreducibility criterion holds over F if and only if there are no finite-dimensional division algebras over F except F itself. Do there exist such fields which are not algebraically closed?

Problem 5.3. Prove that a finitely generated torsion nilpotent group is finite.

In Problems 6.1-6.3 all groups under consideration are subgroups of $GL_n(F)$ (with F algebraically closed) with Zariski topology.

Problem 6.1. Prove that G is connected if and only if G is irreducible (as topological space). By definition, a topological space is irreducible if it cannot be written as a union of two proper (not necessarily disjoint) closed subsets. This has nothing to do with irreducibility of the action of G on V.
Oultine: The backward direction is obvious. For the forward direction, assume that G is connected. Since G is a Noetherian topological space, it is the union of finitely many irreducible components $G_1, \ldots, G_k$ (by definition this means that G_i's are closed irreducible subsets of G which do not contain each other, and it is known that G_i's are unique up to permutation). Show that G naturally acts on the set $\{G_1, \ldots, G_k\}$ by left multiplication and that the stabilizer of G_1 is a closed subgroup of finite index in G. Then deduce that $G_1 = G$.

Problem 6.2. Prove that if G and H are connected subgroups, then the set $GH = \{gh : g \in G, h \in H\}$ is connected. Deduce that the subgroup generated by a family of connected subgroups is connected.

Problem 6.3. Let $H \subseteq G$ be subgroups of $GL_n(F)$, and let $\overline{H}$ be the Zariski closure of H in $GL_n(F)$. Note that $\overline{H} \cap G$ is the Zariski closure of H in G. Prove that

(i) $\overline{H}$ is a subgroup;

(ii) if H is solvable of length k, then $\overline{H}$ is also solvable of length k;

(iii) if H is connected, then $\overline{H} \cap G$ is connected;

(iv) if H is normal in G, then $\overline{H} \cap G$ is normal in G.

Note that using Problems 6.1-6.3 one can justify the definition of the solvable radical of an algebraic group from Lecture 6. Recall that given an algebraic group G, we defined $R(G)$ as the subgroup generated by all connected solvable normal subgroups of G. We claimed that $R(G)$ is itself connected, solvable and normal, and moreover that $R(G)$ is algebraic (that is, Zariski-closed).
First note $R(G)$ is normal by construction and connected by Problem 6.2. To prove that it is solvable we can appeal to the following theorem of Zassenhaus:

Zassenhaus Theorem: *If S is solvable subgroup of $GL_n(F)$, then the solvability length of S is bounded above by a function of n.*

However, there is a way around it. First, by Problem 6.3(i),(iii) and (iv), we can replace every subgroup in the generating set for $R(G)$ by its Zariski closure (which is contained in G since G is algebraic), so $R(G)$ is equal to the subgroup generated by all **algebraic** connected solvable normal subgroups of G. Next we can use a standard result from dimension theory of affine varieties:

Dimension Theorem: *If X and Y are irreducible affine varieties with $X \subseteq Y$ and $X \neq Y$, then* $\dim X < \dim Y$.

Algebraic subgroups of $GL_n(F)$ can naturally be considered as affine varieties of dimension $\leq n^2$. So by Dimension Theorem and Problems 6.1, 6.2 and 6.3(i)(iii), if A and B are distinct algebraic connected solvable normal subgroups of G, then $\overline{AB}$ is also an algebraic connected solvable normal subgroup of G of dimension strictly larger than $\max\{\dim A, \dim B\}$. Thus, if we take R to be an algebraic connected solvable normal subgroups of G of maximal possible dimension, it must contain any other group with this property and thus must equal $R(G)$. This also shows that $R(G)$ is algebraic.

Using Zassenhaus Theorem and repeating the above reasoning, one can show that every linear (not necessarily algebraic) group contains the largest normal solvable subgroup $Solv(G)$ (which contains any other subgroup with this property). One can the define $R(G)$ as the connected component of identity in $Solv(G)$. Note that $Solv(G)$ is sometimes also called the solvable radical of G.

Problem 7.1. Let V be a finite-dimensional vector space and $G \subseteq GL(V)$. Prove that if $\chi_1, \ldots, \chi_m$ are distinct characters of G and $V_{\chi_1}, \ldots, V_{\chi_m}$ the corresponding weight subspaces, then the sum $V_{\chi_1} + \ldots + V_{\chi_m}$ is direct.

Problem 7.2. Prove that if $G \subseteq GL(V)$ is a connected subgroup (with respect to Zariski topology), then $[G, G]$ is also connected. **Hint:** First prove that for a fixed $g \in G$, the map $x \mapsto xgx^{-1}$ from G to G is continuous and deduce that the conjugacy class of any element of G is connected.

Problem 7.3. By a theorem of Maltsev stated at the end of Lecture 7, any solvable linear group $G \subseteq GL_n(F)$, with F algebraically closed, has a normal triangularizable subgroup of finite index $\leq f(n)$ for some function $f : \mathbb{N} \to \mathbb{N}$. Deduce the Zassenhaus theorem stated above from Maltsev's theorem.

Problem 7.4. Prove the following theorem of Platonov: if F is an algebraically closed field of characteristic zero, then any virtually solvable subgroup $G \subseteq GL_n(F)$ has a normal triangularizable subgroup of finite index bounded by a function of n. **Note:** The proof of this theorem is given, for instance, in Appendix B of the following paper:

http://arxiv.org/pdf/1005.1881v1.pdf
which by the way is directly related to part 3 of our class. The proof given there uses the fact that a reductive algebraic group is a torus, which is a basic result in the theory of algebraic groups, but is not so easy to prove from scratch. However, one can avoid using this fact by adapting our proof of Lie-Kolchin theorem.

Problem 8.1. Use Problem 7.4 to prove that in characteristic zero Tits alternative holds true for non-finitely generated groups. Also give an example showing that in positive characteristic zero Tits alternative is false (in general) for non-finitely generated groups.

Problem 8.2. Prove the (full version of) Claim 8.1 (in class we made an additional assumption that g is diagonalizable).

Problem 8.3. Prove Claim 8.2.

Problem 8.4. Let $g = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$. Find $f \in SL_2(\mathbb{Q})$ and $k \in \mathbb{N}$ such that g^k and fg^kf^{-1} generate a free subgroup. Then do the same for $g = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ with an additional assumption that $f \in SL_2(\mathbb{Z})$.

Problem 9.1. Prove Observation 9.0.

Problem 9.2. Let K be a local field, n a positive integer, and let $Dom_n(K)$ be the set of all $n \times n$ matrices over K with dominant eigenvalue. Prove that $Dom_n(K)$ is open in the field topology on $Mat_n(K)$ (the field topology we mean the product topology coming from identification of $Mat_n(K)$ with K^{n^2}).

Problem 10.1. Prove that if a linear group G is Zariski-connected, then $G \times G$ is also Zariski-connected.

Problem 11.1. Prove that if K/E is an arbitrary field extension and n a positive integer, then Zariski topology on E^n is induced from the Zariski topology on K^n under the natural embedding of E^n into K^n.

Problem 11.2. Prove Lemma 11.3: If E is a finitely generated field, n a positive integer, $A \in Mat_n(E)$ and ζ is an eigenvalue of A which is a root of unity (possibly $\zeta \notin E$), then $\zeta^N = 1$ for some N which depends only on E and n. **Hint:** This can be proved in three steps as follows:

(i) Reduce the problem to the case when E is purely transcendental over the prime field E_0 (we used a similar trick in one of the previous lectures).

(ii) Assuming that E is purely transcendental over E_0, prove that $\deg_{E_0}(\zeta) \leq n$.

(iii) Prove that the (multiplicative) order of ζ is bounded by a function of its degree over E_0. Consider separately the cases of zero and positive characteristic.

The following two problems fill the missing details in the proof of Local Field Lemma (Lemma 11.2).

Problem 11.3. Let $\zeta \in \mathbb{C}$ be an algebraic integer such that $|\zeta| = 1$ and ζ is not a root of unity, and let $m(x)$ be the minimal polynomial of ζ over $\mathbb{Q}$. Prove that $m(x)$ has a root η with $|\eta| \neq 1$.

Problem 11.4. Let E be a finitely generated field of characteristic zero, $\zeta \in E$, $m(x)$ the minimal polynomial of ζ over $\mathbb{Q}$ and η any complex root of $m(x)$. Prove that there exists an embedding $\iota : E \to \mathbb{C}$ such that $\iota(\zeta) = \eta$.