Homework #9. Due Thursday, November 11th

In problems 1 and 2, R is a commutative ring with 1, G is a group, and $D_nG = D_n(G, R)$ is the n^{th} dimension subgroup of G. We denote by $\text{Lie}\,(G)$ the Lie algebra of G with respect to the lower central series $\{\gamma_n G\}$ and by $L_D(G)$ the Lie algebra of G with respect to the dimension series $\{D_nG\}$. By $gr R[G]$ we denote the graded algebra of $R[G]$ with respect to powers of the augmentation ideal I.

1. Let $\iota : L_D(G) \to gr R[G]$ be the map given by $\iota(gD_{n+1}G) = (g-1) + I^{n+1}$ for $g \in D_nG$. Recall that we proved that ι is a Lie ring homomorphism. Prove that $\text{Im}\,(\iota)$ generates $gr R[G]$ as an associative R-algebra (with 1). **Note:** In fact, if S is a generating set of G, then $\{\iota(sD_2G) : s \in S\} = \{(s-1) + I^2 : s \in S\}$ will generate $gr R[G]$.

2. Assume that $\text{char} R = 0$. Recall that in Problem 2 of HW#8 it was proved that $D_nF = \gamma_n F$ if F is free.

(a) Recall that there is natural map $\varphi : \text{Lie}\,(G) \to L_D(G)$. Prove that φ is surjective. **Hint:** Choose an epimorphism $\pi : F \to G$, where F is free, and consider the following diagram

$$\begin{array}{ccc} \text{Lie}\,(F) & \longrightarrow & L_D(F) \\ \downarrow & & \downarrow \\ \text{Lie}\,(G) & \longrightarrow & L_D(G) \end{array}$$

Show that it is commutative and the vertical maps are surjective.

(b) Prove that $D_nG \subseteq D_mG \cdot \gamma_n G$ for any $n, m \in \mathbb{N}$. **Hint:** First use (a) to prove this inclusion for $m = n+1$.

(c) Let A be a ring and I an ideal of A. Define the pseudo-norm N on A by $N(u) = 2^{-n(u)}$ where $n(u)$ is the largest integer such that $u \in I^{n(u)}$ (if $u \in \cap_{n\in\mathbb{N}} I^n$, we put $N(u) = 0$). The topology on A induced by this pseudo-norm is called the *I-adic topology.*

Now let $A = R[G]$ and I the augmentation ideal of $R[G]$. Use (b) to prove that $\overline{D_nG} = \overline{\gamma_n G}$ where overline denote the closure in the I-adic topology.

3. Fix $n, d \in \mathbb{N}$, and let $\mathcal{G}(n,d)$ be the class of all groups G which can be generated by d elements and such that $g^n = 1$ for all $g \in G$. Prove that the following statements are equivalent (each of them can be considered as a positive answer to the restricted Burnside problem).

(i) There exists $f(n,d) \in \mathbb{N}$ such that if G is any finite group in $\mathcal{G}(n,d)$, then $|G| \le f(n,d)$

(ii) There exists a finite group G in $\mathcal{G}(n,d)$ such that if H is any other finite group in $\mathcal{G}(n,d)$, then H is a homomorphic image of G.

(iii) Every residually finite group G in $\mathcal{G}(n,d)$ is finite.

Hint: Prove that (ii) "$\Rightarrow$" (i) "$\Rightarrow$" (iii) "$\Rightarrow$" (ii). For the implication (iii) "$\Rightarrow$" (ii) show that if F is the free group of rank d and N is the intersection of the kernels of all homomorphisms $F \to G$, where G is a finite group in $\mathcal{G}(n,d)$, then F/N is a residually finite group in $\mathcal{G}(n,d)$.

4. If G is a finitely generated group and p is a prime, the quotient $G/[G,G]G^p$ is a finite abelian group of exponent p (and thus can be considered as a vector space over $\mathbb{F}_p$). Denote by $d_p(G)$ the dimension of this space (equivalently, $d_p(G) = \log_p[G : [G,G]G^p]$). Recall that $def_p(G)$ denotes the p-deficiency of G.

(a) Prove that $d_p(G) \ge def_p(G) + 1$.

(b) Deduce that if $def_p(G) > -1$, then G has a normal subgroup of index p.

(c) Assume that H is a normal subgroup of G of p-power index. Use Lemma 18.3 to prove that $def_p(H) \ge [G : H] def_p(G)$.

(d) Now assume that $def_p(G) > 0$ and G is finitely presented. Use (a), (c) and Theorem 1.12 in the following paper of M. Lackenby

http://arxiv.org/abs/math/0702571

to prove that G has a finite index subgroup which homomorphically maps onto a non-abelian free group.