Homework #8. Due Thursday, November 4th

1. Let $d \geq 2$, let p be a prime and $G = SL_d^1(\mathbb{F}_p[t])$, the first congruence subgroup of $SL_d(\mathbb{F}_p[t])$. For each $n \geq 1$ let

$$G_n = SL_d^n(\mathbb{F}_p[t]) = \{A \in SL_d(\mathbb{F}_p[t]) : A \equiv I \mod t^n\}.$$

(a) Prove that $\{G_n\}$ is a central series of G.

(b) Let $L(G) = \oplus_{n=1}^{\infty} G_n/G_{n+1}$ be the associated Lie ring. Prove that $L(G)$ is isomorphic to $\mathfrak{sl}_d(t\mathbb{F}_p[t]) = \oplus_{n=1}^{\infty} t^n \mathfrak{sl}_d(\mathbb{F}_p)$ as a graded Lie ring.

Hint for (b): Let $L_n(G) = G_n/G_{n+1}$. Let $\mathfrak{gl}_d(\mathbb{F}_p)$ be the set of all $d \times d$ matrices over $\mathbb{F}_p$ considered as a Lie ring. First, as discussed in class, for each $n \in \mathbb{N}$ define a map $\iota_n : L_n(G) \to \mathfrak{gl}_d(\mathbb{F}_p)$ and show that it is an injective homomorphism of abelian groups. Then use the maps $\{\iota_n\}_{n\in\mathbb{N}}$ to define a grading-preserving injective homomorphism of abelian groups $\iota : L(G) \to \mathfrak{gl}_d(t\mathbb{F}_p[t])$ and show that ι respects the Lie bracket. Finally prove that $\operatorname{Im}\iota = \mathfrak{sl}_d(t\mathbb{F}_p[t])$ (equivalently, $\operatorname{Im}\iota_n = \mathfrak{sl}_d(\mathbb{F}_p)$ for each n). The inclusion $\operatorname{Im}\iota_n \subseteq \mathfrak{sl}_d(\mathbb{F}_p)$ follows form the fact that all elements of G have determinant 1. To prove the opposite inclusion show that $\operatorname{Im}\iota$ contains a subset which generates $\mathfrak{sl}_d(t\mathbb{F}_p[t])$ as a Lie ring.

2. Let R be a commutative ring with 1, G a group, and for $n \in \mathbb{N}$ let $D_nG = D_n(G, R)$ be the n^{th} dimension subgroup of G.

(a) Prove that $\{D_nG\}$ is a central filtration of G (Theorem 16.2(a)).

(b) (corrected) Assume that $\operatorname{char} R = 0$. In class we proved that if G is a finitely generated free group, then $D_nG = \gamma_nG$ for all n. Extend this result to free groups of any rank.

3. Definition:

(i) A Lie (resp. associative) ring L is called *residually finite* if for any nonzero $u \in L$ there is a finite Lie (resp. associative) ring M and a Lie (resp. associative) ring homomorphism $f : L \to M$ such that $f(u) \neq 0$.

(ii) A Lie (resp. associative) ring L is called *hopfian* if any Lie (resp. associative) ring epimorporphism from L to L must be an isomorphism.

(a) Let X be a set and $FA_{\mathbb{Z}}(X)$ the free associative ring on X (that is, $FA_{\mathbb{Z}}(X)$ is the ring of associative polynomials in X with coefficients in $\mathbb{Z}$). Prove that $FA_{\mathbb{Z}}(X)$ is residually finite.

(b) Let A be a residually finite associative ring. Prove that A conisdered as a Lie ring (with bracket $[a, b] = ab - ba$) is also residually finite.

(c) Let X be a set and $FL_{\mathbb{Z}}(X)$ the free Lie ring on X. Use (a), (b) and Theorem 15.2 to prove that $FL_{\mathbb{Z}}(X)$ is residually finite.

(d) Let L be a Lie or associative ring which is finitely generated and residually finite. Prove that L is hopfian. **Hint:** use the same approach as for groups.

(e) Now let X be a finite set and $d = |X|$. Prove that if L is any Lie ring generated by d elements $u_1, \ldots, u_d$ and $\pi : L \to FL_{\mathbb{Z}}(X)$ is an epimorphism, then π is an isomorphism and L is free on $\{u_1, \ldots, u_d\}$. Prove the same for associative rings.

4. Let $X = \{x_1, \ldots, x_d\}$ be a finite set, $F = F(X)$ the free group on X, R a commutative ring with 1, $A = \langle\langle u_1, \ldots, u_d \rangle\rangle$ and $\mu : F \to A^*$ the Magnus representation. Prove that μ is injective.

Hint: For each $n \in \mathbb{N}$ let I_n be the ideal of A generated by $u_1^n, \ldots, u_d^n$. Then for any $n \in \mathbb{N}$ the image of any $f \in F$ under the composition map $\mu_n : F \to A^* \to (A/I_n)^*$ is a polynomial in $u_1, \ldots, u_d$ (that is, you no longer need infinite sums). Compute the largest degree term of that polynomial and conclude that for any $1 \neq f \in F$ there exists $n = n(f) \in \mathbb{N}$ such that $\mu_n(f) \neq 1$.