Homework #7. Due Thursday, October 28th

1. Prove the following version of Ping-pong Lemma for semigroups:

Theorem. *Let G be a group acting on a set X. Assume that there exist subsets X_1 and X_2 of X and elements $a, b \in G$ such that*

$$X_1 \cap X_2 = \emptyset, \quad a(X_1 \cup X_2) \subseteq X_1 \quad \text{and} \quad b(X_1 \cup X_2) \subseteq X_2.$$

Then a and b generate a free semigroup.

2. Realize each of the lamplighter group $\mathbb{Z}\, wr\, \mathbb{Z}$ and $\mathbb{Z}\, wr\, (\mathbb{Z}/n\mathbb{Z})$ as a properly ascending HNN-extension.

3. Let $n \geq 2$, $A \in GL_n(\mathbb{Z})$ and $G = \mathbb{Z} \ltimes_A \mathbb{Z}^n$.

(a) Prove that G is nilpotent if and only if A is unipotent (that is, all eigenvalues of A are equal to 1).

(b) Prove that G is virtually nilpotent if and only if all eigenvalues of A have absolute value 1.

4. Let $n \geq 2$ and G a finitely generated solvable subgroup of $GL_n(\mathbb{Z})$. Assume that for any $g \in G$ all eigenvalues of g have absolute value 1. Prove that some finite index subgroup of G is conjugate in $GL_n(\mathbb{C})$ to a subgroup of $U_n(\mathbb{C})$, the group of $n \times n$ upper-unitriangular matrices over $\mathbb{C}$.
Hint: As a starting point use Lie-Kolchin-Mal'tsev theorem, which we will discuss in a few weeks: *If F is an algebraically closed field, then any solvable subgroup of $GL_n(F)$ has a finite index subgroup which is conjugate to a subgroup of $B_n(F)$, the group of $n \times n$ upper-triangular matrices over F.*

5. If G is a group and A an abelian normal subgroup of G, denote by $\pi : G/A \to \mathrm{Aut}(A)$ the homomorphism given by the conjugation action of G on A. Assume that

(i) $A = \mathbb{Z}^n$ for some n, so $\mathrm{Aut}(A) \cong GL_n(\mathbb{Z})$

(ii) $\pi(G/A)$ is conjugate in $GL_n(\mathbb{C})$ to a subgroup of $U_n(\mathbb{C})$.

(iii) G/A is nilpotent

Prove that G is nilpotent.

6. Let $X = \{x, y\}$ be a set with two elements. Compute all basic commutators in X of degree 6.

7. Let $X = \{x_1, \ldots, x_n\}$ be a set with n elements. For each $k \in \mathbb{N}$ denote by $a_n(k)$ the number of distinct commutators in X of degree k. Prove that

$$a_n(k) = \frac{1}{k} \sum_{d|k} \mu(k/d) n^d.$$

where μ is the Möbius function.

Hint: As discussed in class, $a_n(k)$ is equal to the number of regular words in X of length k. First use Möbius inversion to deduce the formula for $a_k(n)$ from the formula

$$\sum_{d|k} d a_n(d) = n^k \qquad (***).$$

To prove (***) show that if $d \mid k$, then $da_n(d)$ is equal to the number of words of length k which can be written as $w^{k/d}$ where w is aperiodic (that is, w is not a proper power).