Homework #6. Due Thursday, October 14th

1. Let G be a group.

(a) Prove that the following commutator identities hold for any $x, y, z \in G$:

(i) $[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z]$.

(ii) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$.

(iii) $[[x, y], z^x][[z, x], y^z][[y, z], x^y] = 1$

Identity (iii) is called the Hall-Witt identity.

(b) Prove that if A, B and C are normal subgroups of G, then $[A, B, C] \subseteq [B, C, A][C, A, B]$ (where all commutators are left-normed).

(c) Recall that the lower central series $\{\gamma_i G\}$ of G is defined by $\gamma_1 G = G$ and $\gamma_i G = [\gamma_{i-1} G, G]$ for $i \geq 2$. Prove that

$$[\gamma_i G, \gamma_j G] \subseteq \gamma_{i+j} G \text{ for all } i, j \in \mathbb{N}.$$

(d) Now assume that G is finitely generated. Prove that for any $i, j \in \mathbb{N}$ the commutator map $\gamma_i G \times \gamma_j G \to \gamma_{i+j} G$ induces a bilinear map $\gamma_i G/\gamma_{i+1} G \times \gamma_j G/\gamma_{j+1} G \to \gamma_{i+j} G/\gamma_{i+j+1} G$. Deduce that for all $i \in \mathbb{N}$ the quotient $\gamma_i G/\gamma_{i+1} G$ is finitely generated.

2. (a) Let G be a finitely generated nilpotent group. Prove that G is polycyclic and therefore every subgroup of G is finitely generated.

(b) Let G be a group. Let $G = G_1 \supseteq G_2 \supseteq \ldots$ be a *central series* of G, that is, $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \in \mathbb{N}$. Prove that $\gamma_i G \subseteq G_i$ for all i. Now use (a) to prove that if G is finitely generated, then G_i/G_{i+1} is finitely generated for all i.

3. Let G be a finitely generated group.

(a) Let S and S' be two finite generating sets of G. Prove that the growth functions $b_{G,S}(n)$ and $b_{G,S'}(n)$ are equivalent.

(b) Let H be a subgroup of G. Prove that $b_H(n) \preccurlyeq b_G(n)$. Deduce that if G contains a non-abelian free subgroup, then G has exponential growth.

(c) Prove if H is a finite index subgroup of G, then $b_H(n) \sim b_G(n)$.

4. (a) Let G be the Heisenberg group $\mathbb{H}(\mathbb{Z})$. Complete the proof of the fact that $\mathbb{H}(\mathbb{Z})$ has polynomial growth of degree 4.

(b) Let G be the group of upper-unitriangular 4×4 matrices over $\mathbb{Z}$. Prove that G has polynomial growth of degree 10.