Homework #5. Due Thursday, October 7th

1. Prove that for any integer $n \geq 2$ the matrices $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ generate a free group of rank two. **Hint:** Consider the natural action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$ and apply Ping-Pong Lemma with suitable subsets X_1 and X_2 (there exist X_1 and X_2 which work for every $n \geq 2$).

2. Use the isomorphism $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ established in Lecture 9 to prove that

$$SL_2(\mathbb{Z}) = \langle A, B \mid A^4 = 1, A^2 = (AB)^3 \rangle$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. **Hint:** Show that in the group given by the presentation $\langle a, b \mid a^4 = 1, a^2 = (ab)^3 \rangle$ the element a^2 is central and has order 2.

Deduce that $SL_2(\mathbb{Z})$ decomposes as the amalgam $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$.

3. Prove that a group H arises as an HNN-extension if and only if there exists an epimorphism $\pi : H \to \mathbb{Z}$.

4. The purpose of this problem is to prove the formula for the rank of the Cartesian subgroup of the free product of finitely many finite groups (in fact, a minor generalization of this formula). This is done using the notion of rational Euler characterstic which is introduced below.

Let Ω be the smallest class of groups such that

(i) Ω contains the trivial group $\{1\}$ and $\mathbb{Z}$

(ii) Ω is closed under finite direct products

(iii) Ω is closed under finite free products

(iv) Ω is closed under taking finite index subgroups and finite index supergroups

For each group $G \in \Omega$ one can uniquely define the **rational Euler characteristic** $\chi(G) \in \mathbb{Q}$ such that the following properties hold:

(a) $\chi(\{1\}) = 1$ and $\chi(\mathbb{Z}) = 0$

(b) $\chi(G * H) = \chi(G) + \chi(H) - 1$ for any $G, H \in \Omega$

(c) $\chi(G \times H) = \chi(G)\chi(H)$ for any $G, H \in \Omega$

(d) If $G \in \Omega$ and H is a subgroup of index n in G, then $\chi(H) = n\chi(G)$.

The basic idea is that if G is a group which has a finite CW-complex X as its classifying space, then one should have $\chi(G) = \chi(X)$, but then the definition of Euler characteristic has to be extended to a larger class of groups. An explanation of how this can be done is given in the following paper:
C.T.C. Wall, *Rational Euler characteristics.* Proc. Cambridge Philos. Soc. 57 1961 182–184.
Now the actual problem begins. Let $A_1, \ldots, A_k$ be finite groups, set $n_i = |A_i|$, and let F_r be a free group of rank r. Let $G = A_1 * A_2 * \ldots * A_k * F_r$, and let C be the kernel of the natural epimorphism $G \to A_1 \times A_2 \times \ldots \times A_k$ (which sends F_r to $\{1\}$).

(a) Prove that C is free.

(b) Use Euler characteristic to prove that

$$rk(C) = \prod_{i=1}^{k} n_i(r + k - 1 - \sum_{i=1}^{k} \frac{1}{n_i}) + 1.$$

(c) Now assume that $k = 2$ and $r = 0$. Prove that C is freely generated by the set $\{[a, b] : a \in A_1 \setminus \{1\}, b \in A_2 \setminus \{1\}\}$.