Homework #4. Due Thursday, September 30th

1. In class we defined $\mathrm{Aut}^+(F_n)$ to be the subgroup of $\mathrm{Aut}(F_n)$ generated by the elements R_{ij} and L_{ij}.

(i) Prove that $\mathrm{Aut}^+(F_n)$ is a subgroup of index 2 in $\mathrm{Aut}(F_n)$.

(ii) Deduce that $\mathrm{Aut}^+(F_n)$ is the full preimage of $SL_n(\mathbb{Z})$ under the canonical map $\pi : \mathrm{Aut}(F_n) \to GL_n(\mathbb{Z})$. This implies that the definition of $\mathrm{Aut}^+(F_n)$ does not depend on the choice of a free generating set X for F_n (initially this is not obvious, since the definition of the automorphisms R_{ij} and L_{ij} does depend on the choice of X).

(iii) Assume that $n \geq 3$. Prove that $\mathrm{Aut}(F_n)$ is perfect, that is, equal to its commutator subgroup.

2. In class we outlined a proof of Theorem 8.3 which asserts that the natural map $\overline{\pi} : \mathrm{Out}\,(F_2) \to GL_2(\mathbb{Z})$ is an isomorphism. It was left to do be checked that

(i) $GL_2(\mathbb{Z})$ has certain presentation by generators and relations;

(ii) certain elements of $\mathrm{Out}\,(F_2)$ satisfy the relations from that presentation.

One possible way to establish the desired presentation of $GL_2(\mathbb{Z})$ in (i) is to start with the standard presentation of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1\}$ (as we will show next week, $PSL_2(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$), and then use it to first construct a presentation for $SL_2(\mathbb{Z})$ and then finally for $GL_2(\mathbb{Z})$. However, the transition from $SL_2(\mathbb{Z})$ to $GL_2(\mathbb{Z})$ is not necessary to prove Theorem 8.3, that is, one can adapt the argument and work directly with a presentation of $SL_2(\mathbb{Z})$.

Indeed, by Problem 1(ii) the kernel of $\overline{\pi}$ is contained in $\mathrm{Out}^+(F_2)$, so to prove Theorem 8.3 it suffices to check that $\overline{\pi}$ restricted to $\mathrm{Out}^+(F_2)$ is an isomorphism.

Next week (possibly in Homework#5) we (or you) will show that

$$SL_2(\mathbb{Z}) = \langle A, B \mid A^4 = 1, A^2 = (AB)^3 \rangle$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Now define $\alpha, \beta \in \mathrm{Aut}(F_2)$ by $\alpha : x \mapsto y, y \mapsto x^{-1}$ and $\beta : x \mapsto x, y \mapsto xy$ (here $\{x, y\}$ is a fixed free generating set for F_2). Let $\overline{\alpha}$ and $\overline{\beta}$ be the images of α in β in $\mathrm{Out}^+(F_2)$. Show by direct computation that $(\overline{\alpha})^4 = 1$ and $(\overline{\alpha})^2 = (\overline{\alpha}\overline{\beta})^3$. By the same argument as in class, this will finish the proof of Theorem 8.3.

3. Recall again that for any $n \geq 2$ there is a natural epimorphism $\mathrm{Aut}(F_n) \to GL_n(\mathbb{Z})$. The goal of this problem is to show that $\mathrm{Aut}(F_n)$ has a finite index subgroup which homomorphically maps onto $GL_{n-1}(\mathbb{Z})$.

In parts (i)-(iii) below F is a finitely generated free group and K is a finite index normal subgroup of F.

(i) Define $\mathrm{Aut}_K(F)$ to be the subgroup of $\mathrm{Aut}(F)$ consisting of all automorphisms φ which send K to K and act trivially on F/K (that is, $\varphi(K) = K$ and $\varphi(f)f^{-1} \in K$ for any $f \in F$). Prove that $\mathrm{Aut}_K(F)$ is a finite index subgroup of $\mathrm{Aut}(F)$. Subgroups of the form $\mathrm{Aut}_K(F)$ are called *congruence subgroups* of $\mathrm{Aut}(F)$.

(ii) For each $f \in F$ let $\iota_f : x \to x^f = f^{-1}xf$ be the corresponding inner automoprhism of F. Prove that if $[F, F] \subseteq K$, then $\iota_f \in \mathrm{Aut}_K(F)$.

(iii) Note that $\mathrm{Aut}_K(F)$ naturally acts by automorphisms on K and also on $K/[K, K]$. Thus, if $m = rank(K)$, then $\mathrm{Aut}_K(F)$ acts on $\mathbb{Z}^m$ by invertible linear transformations. Now fix $f \in F$ and $\lambda = \pm 1$, and let $V_\lambda(f) = \{u \in \mathbb{Z}^m : \iota_f(u) = \lambda u\}$; clearly $V_\lambda(f)$ is an abelian group. Prove that $V_\lambda(f)$ is $\mathrm{Aut}_K(F)$-invariant.

(iv) Now let $n = rank(F)$, fix a free generating set $X = \{x_1, \ldots, x_n\}$ of F, and let K be the unique subgroup of index 2 in F which contains $x_1, x_2, \ldots, x_{n-1}$. Prove that $V_{-1}(x_n)$ is a free abelian subgroup of rank $n - 1$ with basis $\{\overline{x_i^{x_n} - x_i} : 1 \leq i \leq n - 1\}$ (here overline denotes the image in the abelianization).

(v) Let F and K be as in (iv). Use (iii) and (iv) to prove that $\mathrm{Aut}_K(F)$ homomorphically maps onto $GL_{n-1}(\mathbb{Z})$. **Extra:** What is the index of $\mathrm{Aut}_K(F)$ in this case?

4. Nielsen reduction theorem (Theorem 7.2) yields a general algorithm which, given an n-tuple of elements of F_n, decides whether these elements generate

F_n or not. In the case $n = 2$ one can answer this question almost immediately using the following commutator test.

Theorem (Commutator test)**.** *Let $\{x, y\}$ be a free generating set of F_2, and take any $u, v \in F_2$. Then u and v generate F_2 if and only if the commutator $[u, v] = u^{-1}v^{-1}uv$ is conjugate (in F_2) to $[x, y]$ or $[y, x] = [x, y]^{-1}$.*

(a) Prove the 'only if' ($\Rightarrow$) part of the commutator test. **Hint:** Nielsen transformations.

(b) Now think of how you would prove the 'if' part. I do not know of a nice short algebraic argument. One possible proof is outlined in the following paper of Shpilrain (see Proposition 2.4):

http://www.sci.ccny.cuny.edu/∼ shpil/test1.ps

This is another possible topic for an end-of-semester presentaiton.