Homework #3. Due Thursday, September 23rd

1. Let A and B be groups, and let $A\,wr\,B$ be their wreath product. Recall that

$$A\,wr\,B = A \ltimes (\oplus_{a\in A} B)$$

where A acts on $\oplus_{a\in A}B$ by $(x \cdot b)_a = b_{x^{-1}a}$ for any $b \in \oplus_{a\in A}B$ and $x, a \in A$ (here c_a is the a^{th} component of an element $c \in \oplus_{a\in A}B$). It is common to call $\oplus_{a\in A}B$ the *passive subgroup* of $A\,wr\,B$ and A the *active subgroup*.

(a) Let S be a generating set for A and T a generating set for B. Given $a \in A$, let B_a be the copy of B inside $\oplus_{a\in A}B$ indexed by a, and let $T_a \subseteq B_a$ be the corresponding copy of T. Prove that if we fix $x \in A$, then $T_x \cup S$ is a generating set for $A\,wr\,B$. Deduce that the wreath product of finitely generated groups is finitely generated. **Hint:** First show that $\cup_{a\in A}T_a \cup S$ is a generating set for $A\,wr\,B$.

(b) Prove that $A\,wr\,B$ is solvable if and only if A and B are solvable.

(c) Prove that $A\,wr\,B$ is polycyclic if and only if B is polycyclic and A is solvable and finite.

2. Let p be a prime, and let G_p be the p-lamplighter group $\mathbb{Z}\,wr\,\mathbb{Z}/p\mathbb{Z}$.

(a) Prove that G_p has the following presentations by generators and relations:

(i) $\langle x, y \mid y^p = 1, [y, y^x] = 1, [y, y^{x^2}] = 1, [y, y^{x^3}] = 1, \ldots\rangle$

(ii) $\langle x, y \mid y^p = 1, [y, x, y] = 1, [y, x, x, y] = 1, [y, x, x, x, y] = 1, \ldots\rangle$

where $a^b = a^{-1}ba$, $[a, b] = a^{-1}a^b = a^{-1}b^{-1}ab$, and left-normed commutators $[a_1, \ldots, a_k]$ are defined inductively by $[a_1, \ldots, a_k] = [[a_1, \ldots, a_{k-1}], a_k]$.

(b) Let $\mathbb{F}_p[t, t^{-1}]$ denote the ring of Laurent polynomials over $\mathbb{F}_p$. Construct a faithful 2-dimensional representation for G_p over $\mathbb{F}_p[t, t^{-1}]$, that is, an embedding (injective homomorphism) of G_p into $GL_2(\mathbb{F}_p[t, t^{-1}])$. **Hint:** Send the passive subgroup $\oplus_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}$ to the upper-unitriangular subgroup of $GL_2(\mathbb{F}_p[t, t^{-1}]$ and the active subgroup $\mathbb{Z}$ to a subgroup of the diagonal subgroup of $GL_2(\mathbb{F}_p[t, t^{-1}]$. Can you replace GL_2 by SL_2?

3. Let G be a finitely generated group.

(a) Prove that for any finite group H there are only finitely many homomorphisms from G to H. **Hint:** A homomorphism from G is completely determined by its values on generators.

(b) Prove that for any $n \in \mathbb{N}$ there are only finitely many normal subgroups of index n in G. Then deduce that G has only finitely many subgroups of index n (this was stated in class as Lemma 6.3).

4. Prove Proposition 6.2: any finitely generated residually finite group is hopfian. **Hint:** Let G be finitely generated and $\varphi : G \to G$ a surjective homomorphism. Then $G/\operatorname{Ker}\varphi \cong G$. Deduce that for any $n \in \mathbb{N}$ there is a bijection between the set of all normal subgroups of index n in G and the set of those normal sugbroups of index n in G which contain $\operatorname{Ker}\varphi$. Then use Problem 3 to show that $\operatorname{Ker}\varphi$ must lie in the intersection of all finite index subgroups of G.