Homework #2. Due Thursday, September 16th, by 4pm

1. Let X be a set and $F = F(X)$ the free group on X. The goal of this problem is to prove that for any non-trivial element $f \in F$ the centralizer $C_F(f)$ is cyclic.
Recall that an element $f \in F$ is called cyclically reduced (with respect to X) if the reduced word representing f is of the form $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ (with $x_i \in X$, $\varepsilon_i = \pm 1$) where $x_n^{\varepsilon_n} \neq (x_1^{\varepsilon_1})^{-1}$. Note that any element of $f \in F$ is conjugate to a cyclically reduced element.

(a) Prove that if $f \in F$ and $n \in \mathbb{N}$, then f^n is cyclically reduced if and only if f is cyclically reduced.

(b) Prove that if $f, g \in F$ and $f^n = g^n$ for some $n \in \mathbb{N}$, then $f = g$.

(c) Now prove that if $f, g \in F$ and $f^n = g^m$ for some $n, m \in \mathbb{N}$, then f and g commute. **Hint:** Apply (b) twice.

(d) Deduce that $C_F(f)$ is cyclic for any $f \in F \setminus \{1\}$.

2. Let $F = F(x, y)$ be the free group on two generators, and realize F as the fundamental group of a graph Γ with 1 vertex and 2 loops. For each of the following subgroups H of F (which already appeared in Homework#1) draw the covering space $\widetilde{\Gamma}$ of Γ constructed in the proof of Theorem 2.1. Also explicitly identify the paths in $\widetilde{\Gamma}$ corresponding to the Schreier generators of H (from the conclusion of Theorem 2.1).

(a) $H = [F, F]$, the commutator subgroup of F

(b) $H = \operatorname{Ker} \pi$ where π is the epimorphism from F onto S_3 (symmetric group on 3 letters) which sends x to (12) and y to (23).

3. Let G be a finitely presented group. Define the deficiency of G, denoted by $def(G)$, to be the supremum of the quantity $|X| - |R|$ where (X, R) runs over all presentations of G by generators and relators.

(a) Prove that $def(G) \leq d(G^{ab})$ where $G^{ab} = G/[G, G]$ is the abelianization of G and, as usual, $d(\cdot)$ denoted the minimal number of generators. In particular, this implies that $def(G) < \infty$.

(b) Let H be a subgroup of G of index n. Prove that $def(H) \geq 1 + (def(G) - 1) \cdot n$.

(c) Let $k = def(G)$, and suppose that $k > 0$. Prove that G has a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ such that $n - m = k$, and for each $1 \leq i \leq k$ and each $1 \leq j \leq m$ the total sum of exponents with which x_i appears in r_j is equal to 0.

4. Let (P) be some property of groups. A group G is called *locally* (P) if any finitely generated subgroup of G has (P). For instance, if G is a direct sum of a countable collection of finite groups, then G is locally finite, but not finite. The additive group of rationals $\mathbb{Q}$ is locally free, but not free.

Construct a non-abelian locally free group which is not free. **Hint:** Let F_2 be a free group on two generators, and choose any injective, non-surjective homomorphism $\phi : F_2 \to F_2$. Use it to construct a group G which is representable as $\cup_{n=1}^{\infty} G_n$ where $G_1 \subset G_2 \subset \ldots$ is a strictly ascending chain of subgroups each which is isomorphic to F_2. Then argue that G is locally free, but not free.