Homework #10. Due Thursday, November 18th

1. Let G be a finitely generated group, and suppose that the quotient $\gamma_n G/\gamma_{n+1}G$ is finite for some n. Prove that $\gamma_m G/\gamma_{m+1}G$ is finite for all $m \geq n$. **Hint:** Use the Lie ring $\mathrm{Lie}(G) = \oplus_{n=1}^{\infty}\gamma_n G/\gamma_{n+1}G$ and some facts we proved in Lecture 16.

2. Let K be a field, U a finite set, R a subset of $K\langle\langle U\rangle\rangle$ consisting of elements of positive degree, I the ideal of $K\langle\langle U\rangle\rangle$ generated by R and $A = K\langle\langle U\rangle\rangle/I$. Let M be the ideal of $K\langle\langle U\rangle\rangle$ generated by U, let A_n be the image of M^n in A, and let $a_n = \dim_K A_n/A_{n+1}$.

(a) Prove that $a_{n+m} \leq a_n a_m$ for all m, n. **Hint:** Show that A_{n+m} is K-spanned by elements of the form uv where $u \in A_n$ and $v \in A_m$.

(b) Deduce that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists (and finite).

(c) Now assume that the Golod-Shafarevich condition holds:

$$1 - dt_0 + H_R(t_0) < 0 \text{ for some } t_0 \in (0, 1).$$

Prove that $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$, so that the sequence $\{a_n\}$ grows exponentially.

3. Let H be a group, R a subset of H and N the normal subgroup of H generated by R. Let K be a field and I_R the ideal of the group algebra $K[H]$ generated by $\{r - 1 : r \in R\}$. Prove that there is a canonical isomorphism

$$K[H/N] \cong K[H]/I_R.$$

4. Let G be a group which has a presentation $\langle X|R\rangle$ with $|X| < \infty$ such that

(i) All elements of R lie in $[F(X), F(X)]$, the commutator subgroup of $F(X)$;

(ii) $|R| < |X|^2/4$.

Prove that G is Golod-Shafarevich (GS) in any characterstic.

5. Let p be a prime.

(a) Let F be a free group. Prove that $\deg_p(f^{p^n}) \geq p^n$ for any $f \in F$ and $n \in \mathbb{N}$ (where $\deg_p$ is defined as in Lecture 20).

(b) Let G be a GS group in characteristic p. Use (a) and an idea introduced in Lecture 18 to prove that G has a quotient G' which is p-torsion (that is, every element has p-power order) and is also a GS group in characteristic p (and hence infinite). Thus any such G' gives a counterexample to the general Burnside problem.