Math 8700. Lie Groups.

Problem Set 8. Due on Thursday, November 21.

1. Show by direct computation that

   (a) the root system of $SO_{2n+1}\mathbb{R}$ is isomorphic to B_n

   (b) the root system of $Sp_{2n} := Sp_{2n}(\mathbb{C}) \cap U_n(\mathbb{C})$ is isomorphic to C_n (see page 30 of the book for the definition of Sp_{2n}).

2. Show that B_n, C_n and D_n are indeed root systems without long calculations and without using the fact that they arise as root systems of some Lie groups.

In Problems 3-5 G is a compact Lie group, T a maximal torus of G, $X^*(T)$ the group of characters of T and $X^*(T)_{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Let $N = N_G(T)$, the normalizer of T and $W = N/T$ the Weyl group. As explained in class, we can canonically identify W with a subgroup of $GL(X^*(T)_{\mathbb{R}})$.

3. Let $G = U_n(\mathbb{C})$ and let T be the maximal torus of G consisting of diagonal matrices in G. Compute the normalizer $N = N_G(T)$ and deduce that the Weyl group $W = N/T$ is isomorphic to the symmetric group S_{n-1}. Then show by direct computation that W considered as a subgroup of $GL(X^*(T)_{\mathbb{R}})$ is generated by the reflections $\{s_\alpha\}_{\alpha\in\Phi}$ where $\Phi = \Phi_T(G)$ is the root system of G.

4. Let $w \in W$. Prove that w considered as an element of $GL(X^*(T)_{\mathbb{R}})$ permutes elements of Φ. Then prove that the adjoint action of w on $\mathfrak{g}_{\mathbb{C}}$ sends the roots subspace $\mathfrak{X}_\alpha$ to $\mathfrak{X}_{w(\alpha)}$ for every $\alpha \in \Phi$.

5. Let $\alpha \in \Phi$ and $w_\alpha \in W$ the element defined in Lecture 22. Let $\mathfrak{t} = \mathrm{Lie}(T)$. Using the fact that the adjoint action of w_α on $\mathfrak{t}$ fixes (pointwise) the codimension 1 subspace $\mathfrak{t}_\alpha$, deduce that the action of w_α on $X^*(T)_{\mathbb{R}}$ fixes (pointwise) a hyperplane.