**Math 8700. Lie Groups.**

**Problem Set 6. Due on Thursday, October 24.**

1. Let $\mathfrak{p}_n$ denote the space of $n \times n$ Hermitian matrices in $Mat_n(\mathbb{C})$ and $P_n^+$ the set of positive-definite Hermitian matrices in $GL_n(\mathbb{C})$. In class we verified that $\exp : \mathfrak{p}_n \to P_n^+$ is a continuous bijection. Use the exercise formulated in class to show that exp is actually a homeomorphism. **Hint:** Use the fact that every $A \in \mathfrak{p}_n$ is unitarily conjugate to a diagonal matrix with real entries.

2. The goal of this problem is to finish computation of the fundamental groups for the groups $SL_n(\mathbb{C})$, $(GL_n(\mathbb{R}))^o$, $SL_n(\mathbb{R})$ and $SO_n(\mathbb{C})$ started in class. Below $H$ denotes one of those groups. Recall that in class we showed that any $g \in G = GL_n(\mathbb{C})$ can be uniquely written as $g = pu$ where $p \in P_n^+$ and $u \in U_n(\mathbb{C})$.

   (i) Prove that if $g \in H$, then $p \in H$ and $u \in H$ (we verified this in class for $G = SL_n(\mathbb{C})$).

   (ii) Prove that $H \cap U_n(\mathbb{C}) = SU_n(\mathbb{C})$ if $H = SL_n(\mathbb{C})$ and $H \cap U_n(\mathbb{C}) = SO_n(\mathbb{R})$ in all other cases.

   (iii) Let $\log : P_n^+ \to \mathfrak{p}_n$ be the inverse of the exponential map exp. Prove that $\log(P_n^+ \cap H)$ is an $\mathbb{R}$-subspace of $\mathfrak{p}_n$ and compute this subspace explicitly in each case.

   (iv) Write down the final answer for $\pi_1(H)$ in each case.

3. Let $f : H \to G$ be a Lie group homomorphism. Prove that if $f_* : T_1 H \to T_1 G$ is injective, then for any $h \in H$ the map $f_{*,h} : T_h H \to T_{f(h)} G$ is also injective. **Hint:** Write the homomorphism equation $f(xy) = f(x)f(y)$ as equality of the form $\varphi_1 \varphi_2 = \varphi_3 \varphi_4$ for some maps $\varphi_1, \ldots, \varphi_4$ and take the differential of both sides to relate $f_{*,h}$ with $f_*$.

4. Find isomorphic subalgebras $L_1$ and $L_2$ of $\mathfrak{sl}_2(\mathbb{R})$ such that the corresponding subgroups $\langle \exp(L_1) \rangle$ and $\langle \exp(L_2) \rangle$ of $SL_2(\mathbb{R})$ are not isomorphic.

5. Compute a maximal torus in each of the following groups: $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $U_n(\mathbb{C})$, $SU_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $SO_n(\mathbb{R})$.