Math 8700. Lie Groups.

Problem Set 5. Due on Thursday, October 17.

1. Recall that Theorem 9.3 (= Theorem 8.2 from the book) asserts that for any Lie group G and any $X, Y \in T_1G$ we have $(adX)(Y) = [X, Y]$ where by definition $ad = Ad_*$. The proof of that theorem given in the book shows that $[X, Y]$ considered as a local derivation at 1 is given by the formula

$$[X, Y](f) = \frac{d}{dt}\frac{d}{du} f(e^{tX} e^{uY} e^{-tX}))|_{t=u=0}.$$

Use this formula to show that if $G \xrightarrow{\varphi} H$ is a local homomorphism between Lie groups, then $\varphi_* : T_1G \to T_1H$ is a Lie algebra homomorphism (note that this also provides a new proof of this result in the case of homomorphisms).

2. Let G and H be Lie groups and $L : T_1G \to T_1H$ a Lie algebra homomorphism. Assume that G is connected. Prove that there is at most one Lie group homomorphism $\varphi : G \to H$ such that $\varphi_* = L$. Give an example showing that this may be false if G is not connected. **Hint:** Show that $\varphi(e^A)$ must equal $e^{L(A)}$ for any $A \in T_1G$.

3. Let G be a connected Lie group. Prove that G is abelian if and only if T_1G is abelian.

4. Recall the Campbell-Hausdorff formula proved in class: for any Lie group G there is a neighborhood U of 0 in T_1G such that for any $A, B \in T_1G$ we have $e^A \cdot e^B = e^{Z(A,B)}$ where

$$Z(A, B) = \sum_{n=1}^{\infty} c_n(A, B)$$

and each $c_n(A, B)$ is a homogeneous Lie polynomial of degree n in A and B with coefficients in $\mathbb{Q}$ which does not depend on G. In class we proved that

$$Z(A, B) = A + \left(\int_0^1 \psi(e^{-t\, adB} e^{-adA} - I) dt \right) B,$$

where $\psi(t) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n$. Use this formula to verify that $c_1(A, B) = A + B$, $c_2(A, B) = \frac{1}{2}[A, B]$ and $c_3 = \frac{1}{12}([A, [A, B]] + [B, [B, A]])$.

5. Prove the analogue of the Campbell-Hausdorff formula for the commutator $[e^A, e^B]$ (where for group elements g and h we put $[g,h] = g^{-1}h^{-1}gh$). More specifically, prove that $[e^A, e^B] = e^{Y(A,B)}$ where $Y(A,B) = [A,B] + \sum_{n=3}^{\infty} d_n(A,B)$ with d_n homogeneous of degree n.

6. Let L be a Lie algebra over a field F. Given F-subspaces M and N of L, define $[M,N]$ to be the span of all elements of the set $\{[m,n] : m \in M, n \in N\}$. Define the lower central series $\{\gamma_n L\}$ of L inductively by $\gamma_1 L = L$ and $\gamma_n L = [\gamma_{n-1}L, L]$ for $n \geq 2$. The Lie algebra L is called nilpotent of class c if $\gamma_{c+1}L = \{0\}$ while $\gamma_c L \neq \{0\}$.

(i) For every $c \in \mathbb{N}$ give an example of a Lie algebra of nilpotency class c.

(ii) Let G be a connected Lie group. Use a suitable generalization of the result of Problem 5 to prove that G is nilpotent of class c if and only if $T_1 G$ is nilpotent of class c.

7. Let $\pi : SU_2(\mathbb{C}) \to SO_3(\mathbb{C})$ be the homomorphism defined in class. Prove that π is surjective and $\operatorname{Ker} \pi = \{\pm I_2\}$ (where I_2 is the 2×2 identity matrix).