Math 8700. Lie Groups.

Problem Set 4. Due on Thursday, October 3.

1. Let $g : M \to N$ and $f : N \to P$ be smooth maps between smooth manifolds. Fix $m \in M$ and let $n = g(m)$.

(a) Prove that $(f \circ g)_{*,m} = f_{*,n} \circ g_{*,m}$

(b) Prove that for any smooth curve $\gamma : (a, b) \to M$ and any $\tau \in (a, b)$ we have $(f \circ \gamma)'(\tau) = f_{*,\gamma(\tau)}\gamma'(\tau)$.

2. Let G be a Lie group and $\eta, \gamma : (a, b) \to G$ be smooth curves. Prove that $(\gamma \cdot \eta)'(\tau) = \gamma'(\tau)\eta(\tau) + \gamma(\tau)\eta'(\tau)$ for any $\tau \in (a, b)$ where $\gamma \cdot \eta$ is the pointwise product of γ and η. See Exercise 1.1 in the following online notes for a hint:

`http://www.math.ethz.ch/~salamon/PREPRINTS/liegroup.pdf`

3. Let G be a Lie group, $A \in T_1G$ and let $\gamma_A : \mathbb{R} \to G$ be the unique curve satisfying $\gamma_A(t + s) = \gamma_A(t)\gamma_A(s)$ with $\gamma'(0) = A$. Prove that $\gamma_A(t) = e^{tA}$ for all $t \in \mathbb{R}$.

4. Let $\Phi : G \to H$ be a Lie group homomorphism and $\varphi = \Phi_* : T_1G \to T_1H$ the induced homomorphism of Lie algebras. Prove that $\Phi(e^A) = e^{\varphi(A)}$ for all $A \in T_1G$.

5. Fill in the details of the proof of Proposition 8.2 from the book (= Theorem 9.3 from class). Below is the general plan of the proof. Let $X, Y \in T_1G$ and $ad = Ad_*$

(a) Prove that $(adX(Y))(f) = \frac{d}{dt}\frac{d}{du}f(e^{tX}e^{uY}e^{-tX}))|_{t=u=0}$ for any $f \in O_1(G)$.

(b) Prove that the expression on the right-hand side of (a) is equal to $\frac{d}{du}\frac{d}{dt}f(e^{tX}e^{uY})|_{t=u=0} - \frac{d}{dt}\frac{d}{du}f(e^{uY}e^{tX})|_{t=u=0}$

(c) Let D_X and D_Y be the left-invariant derivations corresponding to X and Y, respectively. Prove that the expression in (b) is equal to $([D_X, D_Y]f)(1)$.

6. Find an example of a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}l_2(\mathbb{C})$ such that the set $exp(\mathfrak{g})$ is not closed.