Math 8700. Lie Groups.
Problem Set 3. Due on Thursday, September 26.

1. Let F be a field and A a (not necessarily associative) F-algebra. Prove that $Der(A)$, the set of F-derivations of A, is a Lie algebra over F with Lie bracket given by $[D_1, D_2] = D_1D_2 - D_2D_1$.

2. Let G be a Lie group and $X \in T_1G$. Prove that the formula $X_g = (L_g)_*X$ for $g \in G$ defines a left-invariant vector field on G. **Warning:** Make sure to check that the correspondence $g \mapsto X_g$ is smooth.

3. The goal of this problem is to formalize an algebraic trick used in the proof of Theorem 7.4. Let M be a smooth manifold and let A be a finite-dimensional $\mathbb{R}$-algebra. Choose a basis $\beta_1, \ldots, \beta_n$ of A. Then for any function $f : M \to A$ there exist (uniquely defined) functions $f_1, \ldots, f_n : M \to \mathbb{R}$ such that $f(m) = \sum_{i=1}^n f_i(m)\beta_i$ for all $m \in M$. By definition, $f \in C^\infty(M, A)$ if and only if each $f_i \in C^\infty(M)$ (check that this definition is independent of the choice of basis).
Now given $D \in \mathrm{Der}C^\infty(M)$, define $\iota_D : C^\infty(M, A) \to C^\infty(M, A)$ by

$$(\iota_D f)(m) = \sum_{i=1}^n (Df_i)(m)\beta_i.$$

Prove that the map $D \mapsto \iota_D$ sends $\mathrm{Der}C^\infty(M)$ into $\mathrm{Der}C^\infty(M, A)$ (in fact, this map is a monomorphism of Lie algebras).

4. Compute explicitly the space of left-invariant derivations of $C^\infty(\mathbb{R}^n)$.

5. Let G and H be Lie groups, $\varphi : G \to H$ a Lie group homomorphism (that is, φ is a smooth map which is also a group homomorphism), and let $\varphi_* : T_1G \to T_1H$ be the induced map on tangent spaces.

(i) Given $A \in T_1G$, let D_A be the left-invariant derivation of $C^\infty(G)$ canonically corresponding to A (as defined in class). Verify that D_A is given by the formula $D_Af(g) = A(f \circ L_g)$. We will use the same notation for left-invariant derivations corresponding to elements of T_1H.

(ii) Again let $A \in T_1G$. Prove that $(D_{\varphi_*A}(f)) \circ \varphi = D_A(f \circ \varphi)$ for any $f \in C^\infty(H)$.

(iii) Use (i) and (ii) to prove that φ_* is a Lie algebra homomorphism, that is, $\varphi_*([A, B]) = [\varphi_*A, \varphi_*B]$ for all $A, B \in T_1G$.