Math 8700. Lie Groups.

Problem Set 2. Due on Thursday, September 19.

1. The goal of this problem is to fill in the details of the proof of Corollary 4.1 that were skipped in class. Let G be a topological group.

(a) Suppose that G is generated (as an abstract group) by a connected subset U containing 1_G. Prove that G is connected.
(b) Let H be a subgroup of G containing a non-empty open subset. Prove that H is open.

Notations. Given a real Lie algebra $\mathfrak{g}$, its complexification is the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ which as a complex vector space is equal to $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and the Lie bracket is defined by the condition $[u \otimes \lambda, v \otimes \mu] = [u, v] \otimes \lambda\mu$, for $u, v \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{C}$.
Consider the following Lie algebras:

(i) $\mathfrak{sl}_n(F) = \{M \in Mat_n(F) : \mathrm{tr}(M) = 0\}$, $F = \mathbb{R}$ or $\mathbb{C}$
(ii) $\mathfrak{so}_n(F) = \{M \in Mat_n(F) : M^t = -M\}$, $F = \mathbb{R}$ or $\mathbb{C}$
(iii) $\mathfrak{u}_n(\mathbb{C}) = \{M \in Mat_n(\mathbb{C}) : M^* = -M\}$
(iv) $\mathfrak{su}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) \cap \mathfrak{sl}_n(\mathbb{C})$

2. Prove that

(a) $\mathfrak{sl}_n(\mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{so}_n(\mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{so}_n(\mathbb{C})$;
(b) $\mathfrak{u}_n(\mathbb{C}) \otimes \mathbb{C} \cong \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{su}_n(\mathbb{C}) \otimes \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C})$.

3. Prove that

(a) The Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$ are not isomorphic
(b) The Lie algebras $\mathfrak{su}_2(\mathbb{C})$ and $\mathfrak{so}_3(\mathbb{R})$ are isomorphic
(c) Deduce that the complexifications of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$ are isomorphic.

4. Let M be a smooth n-manifold and m a point of M. Recall that O_m denotes the ring of germs of smooth functions defined on some neighborhood of m and $\Omega_m = \{f \in O_m : f(m) = 0\}$. Prove that Ω_m is the unique maximal ideal of O_m.

5. Let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called *analytic* at a point $P = (a_1, \ldots, a_n) \in U$ if there exists an open neighborhood V of P on which f is given by a (convergent) power series in

$x_1 - a_1, \ldots, x_n - a_n$, that is,

$$f((x_1, \ldots, x_n)) = \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} (x_1 - a_1)^{i_1} \ldots (x_n - a_n)^{i_n}$$

$$\text{for all } (x_1, \ldots, x_n) \in V.$$

Fix P, let $O = O_P$ be the ring of germs of all functions which are analytic at P and $\Omega = \{f \in O_P : f(P) = 0\}$. Prove the analogue of Proposition 5.2 from class in this setting, that is,

(a) $\Omega = \sum_{i=1}^{n} O \cdot (x_i - a_i)$

(b) The images of the functions of $x_1 - a_1, \ldots, x_n - a_n$ in Ω/Ω^2 form a basis of Ω/Ω^2.

Unlike the proof that we gave in class in the case of smooth functions, your argument should require very few calculations.