Math 8700. Lie Groups.

Problem Set 1. Due on Friday, September 13.

1. Prove that if G and H are Lie groups, then their direct product $G \times H$ is also a Lie group.

2. Prove Proposition 2.1 (basic properties of the exponential map).

3. Prove that $SL_n(\mathbb{R})$ is a Lie group directly by definition (without using Cartan's theorem).

4. Each of the following subsets G is a closed subgroup of $GL_m(\mathbb{C})$ for some m and therefore has the natural structure of a Lie group by Cartan's theorem.

(a) $G = GL_n(F)$, $F = \mathbb{R}$ or $\mathbb{C}$
(b) $G = SL_n(F)$, $F = \mathbb{R}$ or $\mathbb{C}$
(c) $G = O_n(F) = \{A \in Mat_n(F) : AA^t = I\} = \{A \in Mat_n(F) : A^t = A^{-1}\}$, $F = \mathbb{R}$ or $\mathbb{C}$
(d) $G = U_n(\mathbb{C}) = \{A \in Mat_n(\mathbb{C}) : AA^* = I\} = \{A \in Mat_n(\mathbb{C}) : A^* = A^{-1}\}$ where A^* is the conjugate transpose of A.
(e) $G = Sp_{2n}(F) = \{A \in Mat_{2n}(F) : AJA^t = J\}$ (again $F = \mathbb{R}$ or $\mathbb{C}$), where J is the block-diagonal matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ (with each block being an $n \times n$ matrix).

In each case

(i) compute $T_1 G$ and determine whether it is a complex Lie subalgebra
(ii) compute the dimension of G as a real Lie group
(ii) determine whether G is connected, and if not, determine G^o, the connected component of G.

5. Next Monday we will prove that if G is a closed subgroup of $GL_n(\mathbb{C})$, then G^o coincides with the subgroup generated by $\exp(T_1 G)$. Give an example where $\exp(T_1 G) \neq G^o$ and also an example where $\exp(T_1 G) = G^o$. **Hint:** use some of the groups from Problem 4 and start by describing possible Jordan canonical forms of the elements of each group.

6. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the division ring of Hamilton quaternions. Identify the subset $\mathbb{R} \oplus \mathbb{R}i$ of $\mathbb{H}$ with $\mathbb{C}$ (complex numbers) in the natural way; note that any $x \in \mathbb{H}$ can then be uniquely written as $a + bj$ for some $a, b \in \mathbb{C}$.

Let $Mat_n(\mathbb{H})$ be the ring of $n \times n$ matrices over $\mathbb{H}$, and note that every $X \in Mat_n(\mathbb{H})$ can be uniquely written as $A+Bj$ with $A, B \in Mat_n(\mathbb{C})$. Prove that the map from $\varphi : Mat_n(\mathbb{H}) \to Mat_{2n}(\mathbb{C})$ given by

$$\varphi(A + Bj) = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

is a ring monomorphism. As usual, $\overline{X}$ is the conjugate of X.
Now define $GL_n(\mathbb{H})$ to be the multiplicative group of $Mat_n(\mathbb{H})$ or, equivalently,

$$GL_n(\mathbb{H}) = \{A \in Mat_n(\mathbb{H}) : \varphi(A) \in GL_{2n}(\mathbb{C})\}$$

(why is this the same definition) and define

$$SL_n(\mathbb{H}) = \{A \in Mat_n(\mathbb{H}) : \varphi(A) \in SL_{2n}(\mathbb{C})\}.$$

Compute the dimensions of $GL_n(\mathbb{H})$ and $SL_n(\mathbb{H})$ as real Lie groups.