Algebra-II, Spring 2021. Midterm #2
due by 11:59pm on Friday Apr 22nd

Directions: Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use the following resources:

(i) the book by Dummit and Foote
(ii) your class notes (including notes from 7751)
(iii) your previous assignments (homeworks and midterms)
(iv) any materials posted on the Math 7751/7752 collab sites and any materials posted on `http://people.virginia.edu/~mve2x/`

The use of any other resources is prohibited and will be considered a violation of the UVA honor code.

Scoring: The exam contains 5 problems, all of which will count towards your score. The first 4 problems are worth 10 points, and the last problem is worth 12 points. Thus, the maximal possible total is 52, but the score of 50 will count as 100%.

Problem 1: Let F be a field.

(a) Let $f(x) \in F[x]$ be a nonzero polynomial and K/F a field extension. Prove that
$$F[x]/(f(x)) \otimes_F K \cong K[x]/(f(x))$$
as F-algebras.

(b) Let L/F be a finite separable extension. Prove that there exists a finite extension K/F such that $L \otimes_F K \cong \underbrace{K \times \ldots \times K}_{n \text{ times}}$ for some n.

Problem 2: DF, Problem 6 on page 582. Make sure to include all the details.

Problem 3: Let F be a field and $f(x) = x^4 + 1 \in F[x]$.

(a) Determine for which characteristic of F $f(x)$ is separable.
(b) Assume that $f(x)$ is separable and irreducible over F, and let K be the splitting field of $f(x)$ over F. Determine the Galois group $\mathrm{Gal}(K/F)$ (the answer should be of the form "$\mathrm{Gal}(K/F)$ is isomorphic to G" where G is a familiar group).
(c) Suppose that $f(x)$ is irreducible over F. Prove first that F is infinite and then that F has characteristic 0.

Problem 4: Let $S = \{n_1, \ldots, n_k\}$ be a finite set of positive integers ≥ 2 none of which is a perfect square, and let $K = \mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_k})$. You are NOT allowed to refer to HW#6.2.

(a) Prove that $K/\mathbb{Q}$ is a Galois extension and $Gal(K/\mathbb{Q}) \cong \mathbb{Z}_2^m$ for some $m \leq k$
(b) Now assume that $n_1, \ldots, n_k$ are pairwise coprime. Prove that K contains at least 2^k distinct subfields L with $[L : \mathbb{Q}] = 2$.
(c) Keep the extra assumption from (b). Use (b) to prove that $[K : \mathbb{Q}] = 2^k$.

Problem 5: Let p be an odd prime, $\omega = e^{2\pi i/p}$, $K = \mathbb{Q}(\omega)$ and M the unique subfield of K with $[M : \mathbb{Q}] = 2$. Let m be a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ and $\zeta = \sum_{i=0}^{(p-3)/2} \omega^{m^{2i}}$. At the end of Lecture 18 notes posted on collab it is proved that $\zeta \in M$ (we did not get to this part during class; read it before starting on this problem).

(a) Prove that $\zeta \notin \mathbb{Q}$ and deduce that $M = \mathbb{Q}(\zeta)$. **Hint:** Assume that $\zeta \notin \mathbb{Q}$ and deduce that $\deg_{\mathbb{Q}}(\omega) < p - 1$, thereby reaching a contradiction.
(b) Prove by direct computation that if $p = 5$, then $M = \mathbb{Q}(\sqrt{5})$.
(c) Let S be the set of all elements of $(\mathbb{Z}/p\mathbb{Z})^\times$ representable as squares. Prove that $\zeta = \sum_{s \in S} \omega^s$.
(d) Prove that $-1 \in S$ if and only if $p \equiv 1 \mod 4$
(e) Prove that $\overline{\zeta} = \zeta$ if $p \equiv 1 \mod 4$ and $\overline{\zeta} = -1 - \zeta$ if $p \equiv 3 \mod 4$. Deduce that $M \subset \mathbb{R}$ if and only if $p \equiv 1 \mod 4$.
(f) Let L be the unique subfield of K with $[K : L] = 2$. As proved in class, $L = K \cap \mathbb{R}$. Now prove that $M \subset \mathbb{R}$ if and only if $p \equiv 1 \mod 4$ just by using this fact and Galois correspondence (do not use an explicit description of M). **Hint:** What is the

relationship between the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ corresponding to L and M depending on p mod 4?