Algebra-II, Spring 2021. Midterm #1

due by 11:59pm on Tuesday Mar 16th

Directions: Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use the following resources:

(i) the book by Dummit and Foote
(ii) your class notes (including notes from 7751)
(iii) your previous assignments (homeworks and midterms)
(iv) any materials posted on the Math 7751/7752 collab sites and any materials posted on `http://people.virginia.edu/~mve2x/`

The use of any other resources is prohibited and will be considered a violation of the UVA honor code.

Scoring: The exam contains 5 problems, each of which is worth 12 points. If $s_1 \geq s_2 \geq s_3 \geq s_4 \geq s_5$ are your scores on individual problems in decreasing order, your total will be $s_1+s_2+s_3+s_4+\max\{s_5-8,0\}$. Thus, the maximal possible total is 52, but the score of 50 will count as 100%.

1.

(a) Let R be a commutative domain (with 1) and F the field of fractions of R. Let M and N be F-modules, and let $\varphi : M \to N$ be an isomorphism of R-modules. Prove that φ must be an isomorphism of F-modules.

(b) Give an example of finitely generated $\mathbb{C}$-modules M and N and a map $\varphi : M \to N$ such that φ is an isomorphism of $\mathbb{R}$-modules but not an isomorphism of $\mathbb{C}$-modules.

(c) Let M and N be finitely generated $\mathbb{C}$-modules, and suppose that M and N are isomorphic as $\mathbb{R}$-modules. Prove that M and N are isomorphic as $\mathbb{C}$-modules.

2. Let R be a commutative ring with 1, and let M, N and L be R-modules.

(a) Suppose that L is a quotient module of M. Prove that $L \otimes_R N$ is (isomorphic to) a quotient module of $M \otimes_R N$. You are NOT allowed to use Dummit and Foote for this question (class and online notes should be sufficient).

(b) Suppose that M is finitely generated and N is Noetherian. Prove that the R-module $M \otimes_R N$ is Noetherian. **Hint:** Start with the case when M is a free R-module.

(c) Suppose that L is a submodule of M. Is it always true (for any R, M, N and L) that $M \otimes_R N$ contains a submodule isomorphic to $L \otimes_R N$? Prove or give a counterexample.

3. Let R be a commutative ring (with 1) and let M and N be R-modules.

(a) Give an example showing that $T(M \oplus N)$ need not be isomorphic to $T(M) \otimes T(N)$ as rings. **Hint:** Look for an example where $T(M) \otimes T(N)$ satisfies certain nice algebraic property, while $T(M \oplus N)$ does not.

(b) Prove that $S(M \oplus N)$ is isomorphic to $S(M) \otimes S(N)$ as R-algebras. **Hint:** Use the universal property of symmetric algebras to construct a map in one direction and results from previous homeworks to construct a map in the opposite direction. Then prove that the two maps are mutually inverse.

(c) Now assume that R is a field and M and N are finite-dimensional over R. Prove that $\bigwedge(M \oplus N)$ is isomorphic to $\bigwedge(M) \otimes \bigwedge(N)$ as R-modules but not necessarily as rings.

4. Let V be a finite-dimensional vector space over a field F, and let $T : V \to V$ be an F-linear map. Let m be the number of invariant factors of T.

(a) Prove that $m = 1$ if and only if there exists $v \in V$ such that the smallest T-invariant subspace of V containing v is the entire V.

(b) Assume that $m = 1$ and $a(x) = x^6 - 1$ is the (unique) invariant factor of T. Find the number of distinct T-invariant subspaces of V in each of the following 4 cases: $F = \mathbb{R}$, $F = \mathbb{C}$, $F = \mathbb{F}_2$ (field with 2 elements) and $F = \mathbb{F}_3$.

Hint: Rephrase both (a) and (b) as questions about $F[x]$-modules. In (b), the answers in the 4 cases are all finite and all different.

5. Let F be an algebraically closed field, let $A \in Mat_n(F)$ for some $n \in \mathbb{N}$, and let C be the centralizer of A in $Mat_n(F)$. Prove that

$$dim_F(C) \geq n.$$

Hint: First assume that A is in Jordan canonical form and has just one Jordan block; then consider the case when A is an arbitrary matrix in Jordan canonical form, and finally prove the statement for general A.