Homework Assignment # 8

Plan for the week of Mar 29: The main topic will be Galois correspondence (online lectures 20 and 21, Section 14.2 in DF). At the end of Thu Lecture we will either talk about cyclotomic fields (13.6) or continue talking about finite fields, picking up where we left off in Lecture 14 on Tue, March 23 (online lecture 22, Section 14.3 in DF)

Here and in all future assignments "online" or "online notes" refers to Algebra-II lectures posted on my Spring 2010 Algebra-II webpage

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Problems, due by 11:59pm on Friday, April 2nd.

Problem 1: Let p be a prime, $K = \mathbb{F}_p(s,t)$, the field of rational functions over $\mathbb{F}_p$ in two variables, and let $F = \mathbb{F}_p(s^p, t^p)$. Prove that the extension K/F cannot be generated by a single element.

Problem 2: Let $p_1, \ldots, p_n$ be distinct primes. Let $\alpha_i = \sqrt[p_i]{p_i}$ and $\alpha = \sum_{i=1}^{n} \alpha_i$. Prove that $\mathbb{Q}(\alpha_1, \ldots, \alpha_n) = \mathbb{Q}(\alpha)$. **Hint:** this is somewhat similar to the example we discussed at the end of Lecture 18, but certain details of the proof are different.

Problem 3: Let p be a prime, $n \geq 2$ an integer, $f(x) = x^n - p$, and let $K \subset \mathbb{C}$ be the splitting field for $f(x)$ over $\mathbb{Q}$. Recall that $K = \mathbb{Q}(\sqrt[n]{p}, \omega_n)$ where $\omega_n = e^{2\pi i/n}$.

(a) Describe the elements of $\mathrm{Gal}(K/\mathbb{Q})$ by their actions on $\sqrt[n]{p}$ and ω_n.

(b) Let $G = \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^\times$ where $(\mathbb{Z}/n\mathbb{Z})^\times$ acts on $\mathbb{Z}/n\mathbb{Z}$ by multiplication. Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to a subgroup of G and describe an explicit embedding.

(c) (optional) Under which conditions on n and p is this embedding an isomorphism? You are not asked for necessary and sufficient conditions; just the most general sufficient conditions that you can justify.

Problem 4: Let F be a field, let $f(x) \in F[x]$ be an irreducible (over F) separable polynomial of degree n, and let K be a splitting field of $f(x)$.

(a) Prove that $|\mathrm{Gal}(K/F)|$ is a multiple of n and divides $n!$

(b) Let $n = 3$. Prove that $\mathrm{Gal}(K/F)$ is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or S_3.

(c) Assume that $F = \mathbb{R}$ (real numbers) and $f(x)$ has a non-real root. Prove that $\mathrm{Gal}(K/F)$ is isomorphic to S_3.

Problem 5: Let $f(x)$ and $g(x)$ be irreducible polynomials in $\mathbb{F}_p[x]$ of the same degree and let $F = \mathbb{F}_p[x]/(f(x))$. Prove that $g(x)$ splits completely over F.

Problem 6: Let p be a prime, n a positive integer and $\Phi_n(x) = x^{p^n} - x \in \mathbb{F}_p[x]$. Prove that $\Phi_n(x)$ is equal to the product of all monic irreducible polynomials in $\mathbb{F}_p[x]$ whose degrees divide n (where each polynomial occurs with multiplicity one).