Homework Assignment # 7

Plan for the week of Mar 22: Primitive element theorem (online Lecture 18). Structure of finite fields (see the first page and a half of online Lecture 22). Galois extensions and Galois groups (online lecture 19, Section 14.1 in DF), maybe start Galois correspondence (online lecture 20, Section 14.2 in DF).

Here and in all future assignments "online" or "online notes" refers to Algebra-II lectures posted on my Spring 2010 Algebra-II webpage

> http://people.virginia.edu/~mve2x/7752_Spring2010/

Problems, due by 11:59pm on Friday, March 26th.

Problem 1 (practice): Let F be a field and Ω a subset of $F[x]$. Use the existence and uniqueness of algebraic closures to prove that there exists a unique splitting field for Ω over F up to F-isomorphism. **Hint:** Use the fact that any splitting field K for Ω lies in some algebraic closure of F (this was proved at the beginning of Lecture 12 on March 16).

Problem 2: Let K/F be a field extension, and let K_1 and K_2 be subfields of K containing F such that the extensions K_1/F and K_2/F are normal. Prove that the extensions $K_1 K_2/F$ and $K_1 \cap K_2/F$ are also normal. Here $K_1 K_2$ is the *composite field of K_1 and K_2* (see pp. 528-529 of DF for the definition).

Problem 3: Prove the interesting part of Corollary 18.7 form online notes: if $K/L/F$ is a tower of algebraic extensions and K/L and L/F are separable, then K/F is separable (see online notes for a hint). Give a detailed argument.

Problem 4:

    (a) Prove that if $K/\mathbb{Q}$ is any field extension, then any field automorphism of K must fix $\mathbb{Q}$ elementwise.

    (b) Prove that $\mathrm{Aut}(\mathbb{R})$, the group of field automorphism of $\mathbb{R}$, is trivial. See [DF, Problem 7, p.567] for a sketch of the proof.

Before doing problems 5-7 below read § 13.6 in DF.

Problem 5: Let p be a prime, $n \geq 2$ an integer, $f(x) = x^n - p$, and let $K \subset \mathbb{C}$ be the splitting field for $f(x)$ over $\mathbb{Q}$.

    (a) Prove that $K = \mathbb{Q}(\sqrt[n]{p}, \omega_n)$ where $\omega_n = e^{2\pi i/n}$.

    (b) Prove that $[K : \mathbb{Q}] \leq n\varphi(n)$ where φ is the Euler function.

(c) Assume that n is prime. Prove that inequality in (a) is equality.

(d) Let $p = 3$ and $n = 12$. Prove that inequality in (a) is strict and find $[K : \mathbb{Q}]$. **Hint:** Compute ω_{12} explicitly.

Problem 6: Problem 4 in [DF, p.555]

Problem 7: Problem 5 in [DF, p.555]