Homework Assignment # 5

Plan for the week of Mar 1: Field extensions and algebraic closures (13.1, 13.2, 13.4 and online lectures 14, 15).

Here and in all future assignments "online" refers to Algebra-II lectures posted on my Spring 2010 Algebra-II webpage

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Problems, due by 11:59pm on Friday, March 5th.

Problem 1: Find the number of distinct conjugacy classes in the group $GL_3(\mathbb{F}_2)$ (where $\mathbb{F}_2$ is the field with 2 elements) and specify one element in each conjugacy class.

Problem 2:

(a) Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing that this does not hold for 4×4 matrices.

(b) A matrix A is called idempotent if $A^2 = A$. Prove that two idempotent $n \times n$ matrices are similar if and only if they have they same rank. **Hint:** What is the minimal polynomial of an idempotent matrix? How does rank relate to eigenvalue 0?

Problem 3: Prove that there is no matrix $A \in Mat_{10}(\mathbb{Q})$ satisfying $A^4 = -Id$.

Problem 4: DF, Problem 15 on page 500.

Problem 5: DF, Problem 20 on page 501. Also find an explicit $P \in GL_n(F)$ such that $A = PBP^{-1}$ where A and B are the two matrices in the form (you may choose which one is A and which one is B).

Problem 6: Let V be an n-dimensional vector space over some field F, and let $T : V \to V$ be a **nilpotent** F-linear map. Prove that $T^n = 0$ in two different ways:

(a) using JCF

(b) without using JCF or RCF, but instead looking at the sequence of kernels $\{\text{Ker}\,(T^k)\}_{k=1}^{\infty}$. The idea here is similar to Problem 2 in HW#4 of Algebra 1.

Problem 7: Given $A \in Mat_n(\mathbb{C})$, define its exponential $\exp(A)$ as in DF, Problem 41(b) on page 503 (you do not have to prove convergence of the series defining $\exp(A)$).

(a) Prove that if $A = PBP^{-1}$ for some $P \in GL_n(\mathbb{C})$, then $\exp(A) = P \exp(B)\, P^{-1}$

(b) Use (a) to prove that $\det(\exp(A)) = \exp(\mathrm{tr}(A))$.