Homework Assignment # 4. Preliminary version

Plan for the week of Feb 22: Finish Rational Canonical Form (12.2, online lectures 10-11), Jordan Canonical Form (12.3, online lecture 12).

Here and in all future assignments "online" refers to Algebra-II lectures posted on my Spring 2010 Algebra-II webpage

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

Problems, due by 11:59pm on Friday, February 26th.

Problem 1. Let $R = \mathbb{R}[x]$, $F = R^3$ (the standard 3-dimensional R-module) and N the R-submodule of F generated by $(1 - x, 1, 0)$, $(-2, 4 - x, 0)$ and $(1, -5, -x)$.

(a) Find compatible bases for F and N, that is, bases satisfying the conclusion of the compatible bases theorem (AKA submodule structure theorem). **Note:** an algorithm for computing such bases is given in Lecture 8.

(b) Describe the quotient module F/N in IF and ED forms.

Problem 2. Let R be a PID. For an R-module M denote by $d(M)$ the minimal number of generators of M (this quantity was called the rank in HW#3).

(a) Prove that if M is a finitely generated R-module and N is a submodule of R, then $d(N) \leq d(M)$

(b) Let $a \in R$ be a nonzero non-unit. Find (with proof) the number of submodules of R/aR in terms of the prime decomposition of a.

Problem 3. DF, Problem 4, page 469. Warning: the definition of rank in this problem (and in DF in general is different from HW#4. Then give an example showing that the assertion of this problem would be false if we replace $rk(U)$ by $d(U)$ for every module U in the problem.

Problem 4. DF, Problem 6, page 488.

Problem 5. Given a matrix A we denoted by $\chi_A(x)$ its characteristic polynomial. Determine the number of possible RCFs of 8×8 matrices A over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$. Explain your argument in detail.