Homework Assignment # 3.

Plan for the week of Feb 15: Classification of finitely generated modules over PIDs (12.1, online lecture 9). Rational Canonical Form (12.2, online lectures 10-11).

Here and in all future assignments "online" refers to Algebra-II lectures posted on my Spring 2010 Algebra-II webpage

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

Problems, due by 11:59pm on Friday, February 19th.

Problem 1. In class we briefly discussed a simple characterization of R-modules where $R = \mathbb{Z}$ or $R = F[x]$ for some field F (see online Lecture 1 and DF, pp. 340-341 for more details). More specifically, we explained why

- there is a natural correspondence between $\mathbb{Z}$-modules and abelian groups
- for any field F, there is a natural correspondence between $F[x]$-modules and pairs (V, A) where V is an F-vector space and $A : V \to V$ is an F-linear map.

State and prove similar characterizations for the R-modules in the following two cases:

(a) $R = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$
(b) $R = F[x, y]$ for some field F.

Problem 2. DF, Problem 8 on page 455.

Problem 3*. Prove Theorem N1 from Lecture 8: Let R be a ring (with 1), let M be an R-module and N its submodule. Prove that M is Noetherian $\iff$ N and M/N are both Noetherian.

Problem 4. Let A be a ring (with 1). A subring B of A is called a *retract* if there exists a surjective ring homomorphism $\varphi : A \to B$ such that $\varphi_{|B} = id_B$, that is, $\varphi(b) = b$ for all $b \in B$.
Now let M and N be two R-modules. Prove that the tensor algebra $T(M)$ is (naturally isomorphic to) a subalgebra of $T(M \oplus N)$ and that this subalgebra is a retract. Also prove the analogous statement about the symmetric algebras.

Problem 5: Define the rank of an R-module M, denoted by $\mathrm{rk}(M)$, to be the minimal number of generators (WARNING: this definition is different from the definition in DF; the two definitions coincide for free modules over commutative rings).

(a)* Let R be a PID, M be a finitely generated R-module and $R/a_1R \oplus \ldots \oplus R/a_mR \oplus R^s$ its invariant factor decomposition, that is, $a_1, \ldots, a_m$ are nonzero non-units and $a_1 \mid a_2 \mid \ldots \mid a_m$. Prove that $\mathrm{rk}(M) = m + s$. **Warning:** It is not true in general that $\mathrm{rk}(P \oplus Q) = \mathrm{rk}(P) + \mathrm{rk}(Q)$.

(b) Again let R be a PID. Let F be a free R-module of rank n with basis $e_1, \ldots, e_n$, let N be the submodule of F generated by some elements $v_1, \ldots, v_n \in F$, and let $A \in Mat_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

Find a simple condition on the entries of A which holds if and only if $\mathrm{rk}(F/N) = n$.

Hint for 3: The forward direction is easy. For the backwards direction, observe that if $\{P_i\}$ is an ascending chain of submodules of M, then $\{P_i \cap N\}$ is an ascending chain of submodules of N and $\{(P_i + N)/N\}$ is an ascending chain of submodules of M/N.

Hint for 5(a): Let p be a prime dividing a_1. How is M related to $M' = (R/pR)^{m+s}$ and what is $\mathrm{rk}(M')$ (and why)?