Homework Assignment # 2.

Plan for the week of Feb 8: Modules over PIDs (12.1, online lectures 7-9). At the beginning of Tue class we will probably also briefly discuss a characterization of modules over $F[x]$, F a field (see DF, pp. 340-341 and online lecture 1).

Here and in all future assignments "online" refers to Algebra-II lectures posted on my Spring 2010 Algebra-II webpage

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Problems, due by 11:59pm on Friday, February 12th.

Problem 1. Problem 25 on pp. 377 of DF. Deduce that for any commutative ring R with 1 we have $R[x]\otimes_R R[y] \cong R[x,y]$ as R-algebras (here $R[x]$ and $R[y]$ are isomorphic copies of the ring of polynomials over R in 1 variable and $R[x,y]$ is the ring of polynomials over R in 2 commuting variables).

Problem 2.

(a) Finish the proof of the fact that $A = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ and $B = \mathbb{C}\times\mathbb{C}$ are isomorphic as $\mathbb{C}$-algebras. Recall that in class we constructed a map $\varphi : A \to B$ as a composition of four ring isomorphisms $\varphi_1, \ldots, \varphi_4$. What you have to do is describe each φ_i explicitly and deduce that each φ_i is $\mathbb{C}$-linear (and hence also a $\mathbb{C}$-algebra isomorphism).

(b) Explain why $\{1\otimes 1, 1\otimes i\}$ is a basis for A over $\mathbb{C}$. Now compute $\varphi(1\otimes 1)$ and $\varphi(1\otimes i)$ where φ is an isomorphism from (a) (note that φ is completely determined by its values on a $\mathbb{C}$-basis of A).

(c) Prove that there exist precisely 2 $\mathbb{C}$-algebra isomorphisms from A to B. **Hint:** First prove ≥ 2 and then ≤ 2. See the last page for a more detailed hint.

Problem 3. The main goal of this problem is to classify 2-dimensional $\mathbb{R}$-algebras ($\mathbb{R}$=reals), that is, $\mathbb{R}$-algebras which are 2-dimensional as vector spaces over $\mathbb{R}$.

Let F be a field with $\mathrm{char}(F) \neq 2$, and let A be a 2-dimensional F-algebra with 1.

(a) Let $u \in A$ be any element which is not an F-multiple of 1. Prove that

   (i) u generates A as an F-algebra, that is, the minimal F-subalgebra of A containing u and 1 is A itself.
   (ii) u satisfies a quadratic equation $au^2 + bu + c = 0$ for some $a, b, c \in F$ with $a \neq 0$.

(b) Show that there exists $v \in A$ such that $v^2 \in F$. **Hint:** take any u as in (a), and look for v of the form $u + \beta$ with $\beta \in F$.
(c) Deduce from (b) that A is isomorphic as an F-algebra to $F[x]/(x^2 - c)$ for some $c \in F$.
(d) Prove that if $c = d^2$ for some $d \in F \setminus \{0\}$, then $F[x]/(x^2 - c) \cong F \times F$.
(e) Now let $F = \mathbb{R}$ (real numbers). Prove that in (c) one can choose $c = 0, 1$ or -1. Then prove that the algebras $\mathbb{R}[x]/(x^2 + 1)$, $\mathbb{R}[x]/(x^2 - 1)$ and $\mathbb{R}[x]/(x^2)$ are pairwise non-isomorphic. **Hint:** the algebras can be distinguished from each other by simple abstract properties.

Problem 4. Let V and W be finite dimensional vector spaces over a field F, let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_m\}$ a basis of W.
Let $\varphi : V \otimes_F W \to Mat_{n\times m}(F)$ be the F-linear transformation such that $\varphi(v_i \otimes w_j) = e_{ij}$ where e_{ij} is the matrix whose (i, j)-entry is equal to 1 and all other entries are equal to 0 (note that such transformation exists and is unique because $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes_F W$; furthermore, φ is an isomorphism since matrices $\{e_{ij}\}$ form a basis of $Mat_{n\times m}(F)$).
Prove that for a matrix $A \in Mat_{n\times m}(F)$ the following are equivalent:

(a) $A = \varphi(v \otimes w)$ for some $v \in V, w \in W$ (note: v and w need not be elements of the above bases)
(b) $rk(A) \leq 1$.

Problem 5 (practice). Let $R = \oplus_{n=0}^{\infty} R_n$ be a graded ring. An element $r \in R$ is called *homogeneous* if $r \in R_n$ for some n.

Any $r \in R$ can be uniquely written as $r = \sum_{n=0}^{\infty} r_n$ where $r_n \in R_n$ and only finitely many r_n's are nonzero. The elements $\{r_n\}$ are called the *homogeneous components of* r.
(a) Let I be an ideal of R. Prove that the following are equivalent:

   (i) I is a graded ideal, that is, $I = \oplus_{n=0}^{\infty} I \cap R_n$
   (ii) For each $r \in I$ all homogeneous components of r also lie in I

(b) Let I be an ideal of R generated by homogeneous elements (possibly of different degrees). Prove that I is graded.

Problem 6. Before solving this problem read about the exterior algebras (see DF, § 11.5 and the end of online Lecture 6).

(a) Let R be a commutative ring with 1 and M an R-module. Let $m_1, \ldots, m_k$ be elements of M and $\sigma \in S_k$ a permutation. Prove that $m_{\sigma(1)} \wedge \ldots \wedge m_{\sigma(k)} = (-1)^\sigma m_1 \wedge \ldots \wedge m_k$.

(b) Use (a) to prove Proposition 6.5 from the online Lecture 6.

Extra Hint for Problem 2(c):

- To prove that there are at least 2 $\mathbb{C}$-algebra isomorphisms from A to B show that either A or B has a non-trivial $\mathbb{C}$-algebra automorphism (the assertion is true for both A and B, but you only need to prove it for one of them).
- To prove ≤ 2 find enough restrictions on $\psi(1 \otimes 1)$ and $\psi(1 \otimes i)$ where $\psi : A \to B$ is a $\mathbb{C}$-algebra isomorphism to deduce that there are at most 2 choices for the pair $(\psi(1 \otimes 1), \psi(1 \otimes i))$.