Algebra-II, Spring 2021. Final exam
due by 12pm on Saturday May 15th

Directions: Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use the following resources:

(i) the book by Dummit and Foote
(ii) your class notes (including notes from 7751)
(iii) your previous assignments (homeworks and midterms)
(iv) any materials posted on the Math 7751/7752 collab sites and any materials posted on `http://people.virginia.edu/~mve2x/`

The use of any other resources is prohibited and will be considered a violation of the UVA honor code.

Scoring: The exam contains 6 problems, each of which is worth 15 points. If $s_1 \geq s_2 \geq s_3 \geq s_4 \geq s_5 \geq s_6$ are your scores on individual problems in decreasing order, your total will be $s_1 + s_2 + s_3 + s_4 + s_5 + \max\{s_6 - 10, 0\}$. Thus, the maximal possible total is 80, but the score of 75 will count as 100%.

Problem 1: Let F be an algebraically closed field with $\operatorname{char} F \neq 2$.

(a) Let J be a Jordan block of size n with eigenvalue λ over F. Determine the Jordan canonical form of the matrix J^2. **Hint:** Consider the cases $\lambda = 0$ and $\lambda \neq 0$ separately.
(b) Let $A \in Mat_4(F)$. Determine necessary and sufficient conditions for A to have a square root, i.e. for there to exist a matrix $B \in Mat_4(F)$ such that $A = B^2$. State your answer in the form: A has a square root $\iff JCF(A)$ satisfies certain conditions. Make sure to prove your answer.

Problem 2: Let $f(x) = (x^2 - 2)(x^3 - 3) \in \mathbb{Q}[x]$ and let $K \subseteq \mathbb{C}$ be the splitting field of $f(x)$ over $\mathbb{Q}$.

(a) Prove that $\operatorname{Gal}(K/\mathbb{Q}) \cong S_3 \times \mathbb{Z}/2\mathbb{Z}$
(b) Describe explicitly all subfields L of K with $[L : \mathbb{Q}] = 6$ by giving explicit generators for each subfield and determine the corresponding subgroup of $\operatorname{Gal}(K/\mathbb{Q})$ for each such L. Make sure to prove that you found all such subfields/subgroups.

Problem 3: Let K/F be a finite Galois extension and $G = \mathrm{Gal}(K/F)$.

(a) Assume that G is a simple group, let $\alpha \in K \setminus F$ and $\mu_{\alpha,F}(x)$ the minimal polynomial of α over F. Prove that K is a splitting field for $\mu_{\alpha,F}(x)$.

(b) Let $n = [K : F]$, and fix integers m and l with $ml = n$. Find a condition on the subfield lattice of K/F which is equivalent to the following: G can be written as a semidirect product $G = A \rtimes B$ for some subgroups A and B where $|A| = m$ and $|B| = l$. Make sure to prove your answer.

Note: your answer should be stated in terms of subfields of K/F and the integers m and l; you can refer to properties like 'normal extension' or 'Galois extension', but you cannot mention any groups in your characterization.

Problem 4: In this problem p denotes a fixed prime number. Let P denote the set of all prime numbers. Recall from Lecture 23 that a *supernatural number* is a formal product $\alpha = \prod\limits_{q \in P} q^{a_q}$ where each $a_q \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Note that natural numbers can be identified with supernatural numbers for which all a_q are finite and only finitely many a_q are nonzero. The goal of this problem is to construct a natural bijection between supernatural numbers and subfields of $\overline{\mathbb{F}_p}$ (a fixed algebraic closure of $\mathbb{F}_p$).

For each $i \in \mathbb{N}$ let $F_i = \mathbb{F}_{p^i} \subseteq \overline{\mathbb{F}_p}$. Given a subfield L of $\overline{\mathbb{F}_p}$, define $I(L) = \{i \in \mathbb{N} : F_i \subseteq L\}$.

(a) Prove that for any subfield L of $\overline{\mathbb{F}_p}$ we have $L = \bigcup\limits_{i \in I(L)} F_i$.

(b) Suppose that $I = I(L)$ for some subfield L of $\overline{\mathbb{F}_p}$. Prove that I satisfies the following two conditions:
   (i) I is closed under divisors: for any $i \in I$, any positive divisor of I also lies in i
   (ii) I is closed under least common multiples: for any $i, j \in I$ we have $LCM(i, j) \in I$

(c) Now let I be a subset of $\mathbb{N}$ satisfying conditions (i) and (ii) from (b), and let $L = \bigcup\limits_{i \in I} F_i$. Prove that L is a subfield and that $I(L) = I$ (it is clear that $I(L)$ contains I, but you need to prove the equality!)

(d) Deduce from (b) and (c) that there exists a bijection between subfields of $\overline{\mathbb{F}_p}$ and subsets of $\mathbb{N}$ satisfying (i) and (ii)

(e) Now construct a natural bijection between subsets of $\mathbb{N}$ satisfying (i) and (ii) and supernatural numbers. Your bijection should satisfy two conditions:
(iii) finite subsets of $\mathbb{N}$ correspond to natural numbers
(iv) if I and J are two subsets of $\mathbb{N}$ satisfying (i) and (ii) and $\alpha_I = \prod_{q\in P} q^{a_q}$ and $\beta_I = \prod_{q\in P} q^{b_q}$ are the corresponding supernatural numbers, then $I \subseteq J \iff a_q \leq b_q$ for all $q \in P$.

Problem 5: [DF, Problem 12, page 654(a)-(c)]. Let K be a subfield of $\mathbb{C}$ maximal with respect to the property "$\sqrt{2} \notin K$."

(a) Show that such a field K exists.
(b) Show that $\mathbb{C}$ is algebraic over K. **Hint:** Assume that there exists $\alpha \in \mathbb{C}$ which is transcendental over K and consider the extension $K(\alpha)/K$.
(c1) Let L/K be a finite Galois extension, with $L \subseteq \mathbb{C}$. Prove that $\mathrm{Gal}(L/K)$ is cyclic of order 2^k for some $k \in \mathbb{Z}_{\geq 0}$. **Hint:** What can you say about the subfield lattice of L/K based on the definition of K? Use Galois correspondence to translate this property into a condition on $\mathrm{Gal}(L/K)$, and then show that the only groups satisfying that condition are cyclic groups of 2-power order.
(c2) Now use (c) to prove that any finite extension L/K, with $L \subseteq \mathbb{C}$, is Galois.

Problem 6: Let F be a field, with $\mathrm{char}\,(F) \neq 2$ and

$$A = F[x, y, z]/(x^2 + y^2 + z^2 - 4).$$

For every polynomial $p \in F[x, y, z]$ we denote by $\bar{p}$ the image of p in A. It is easy to see that A is a domain (you need not prove this), so we can consider its field of fractions $K = Frac(A)$.

(a) Let $a = \bar{x} + \bar{y} + \bar{z}$, $b = \bar{x}\bar{y} + \bar{x}\bar{z} + \bar{y}\bar{z}$ and $c = \bar{x}\bar{y}\bar{z}$, and let $L = F(a, b, c)$, the subfield of K generated by F, a, b and c. Prove that K/L is a finite Galois extension and $\mathrm{Gal}(K/L) \cong S_3$.

Hint: First find a faithful action of S_3 on K such that $L \subseteq K^{S_3}$. Then find a cubic polynomial $f(t) \in L[t]$ such that K is a splitting field for f over L. Then deduce (a) from these two results (and some theorems proved in class)

(b) Prove that the extension L/F is purely transcendental and find an explicit transcendence basis for it.