Homework Assignment # 8.

Plan for the next week: Galois extensions and Galois groups (online lecture 19, Section 14.1 in DF), Galois correspondence (online lectures 20 and 21, Section 14.2 in DF).

Problems, to be submitted by Thu, March 28th.

Problem 1: Let p be a prime, $K = \mathbb{F}_p(s,t)$, the field of rational functions over $\mathbb{F}_p$ in two variables, and let $F = \mathbb{F}_p(s^p, t^p)$. Prove that the extension K/F cannot be generated by a single element.

Problem 2: Prove the interesting part of Corollary 18.7 form online notes: if $K/L/F$ is a tower of algebraic extensions and K/L and L/F are separable, then K/F is separable (see online notes for a hint). Give a detailed argument.

Problem 3: Let $p_1, \ldots, p_n$ be distinct primes. Let $\alpha_i = \sqrt[p_i]{p_i}$ and $\alpha = \sum\limits_{i=1}^{n} \alpha_i$. Prove that $\mathbb{Q}(\alpha_1, \ldots, \alpha_n) = \mathbb{Q}(\alpha)$. **Hint:** this is somewhat similar to the example we discussed at the end of Lecture 18, but certain details of the proof are different.

Problem 4: Let p be a prime, $n \geq 2$ an integer, $f(x) = x^n - p$, and let $K \subset \mathbb{C}$ be the splitting field for $f(x)$ over $\mathbb{Q}$. Recall that $K = \mathbb{Q}(\sqrt[n]{p}, \omega_n)$ where $\omega_n = e^{2\pi i/n}$.

(a) Describe the elements of $\mathrm{Gal}\,(K/\mathbb{Q})$ by their actions on $\sqrt[n]{p}$ and ω_n.

(b) Let $G = \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^{\times}$ where $(\mathbb{Z}/n\mathbb{Z})^{\times}$ acts on $\mathbb{Z}/n\mathbb{Z}$ by multiplication. Prove that $\mathrm{Gal}\,(K/\mathbb{Q})$ is isomorphic to a subgroup of G and describe an explicit embedding.

(c) (optional) Under which conditions on n and p is this embedding an isomorphism? You are not asked for necessary and sufficient conditions; just the most general sufficient conditions that you can justify.

Problem 5: Let F be a field, let $f(x) \in F[x]$ be an irreducible (over F) separable polynomial of degree n, and let K be a splitting field of $f(x)$.

(a) Prove that $|\mathrm{Gal}\,(K/F)|$ is a multiple of n and divides $n!$

(b) Let $n = 3$. Prove that $\mathrm{Gal}\,(K/F)$ is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or S_3.

(c) Let $n = 4$ and assume that $|\mathrm{Gal}\,(K/F)| = 8$. Determine the isomorphism class of $\mathrm{Gal}\,(K/F)$.